STABILITY OF DISCRETE TIME SYSTEMS WITH UNILATERAL NONLINEARITIES

J.C. Tsai
Department of Mechanical Engineering
Northwestern University
Evanston, Illinois

J. Edward Colgate
Department of Mechanical Engineering
Northwestern University
Evanston, Illinois

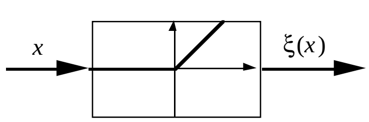
ABSTRACT

This paper considers the effect of unilateral nonlinearities on the stability of discrete time control systems, a problem of some importance in haptic display. The unilateral nonlinearity is a simple piecewise linear function: $\xi(x) = x$ for $x \geq 0$, $\xi(x) = 0$ for $x < 0$. This function plays an important role in the modeling of collisions, and is a part of essentially all implementations of virtual surfaces. The unilateral nonlinearity is, in part, responsible for the instability often seen when haptic display operators contact virtual surfaces.

In this paper, it is shown that an operator contacting a virtual surface via a haptic display can reasonably be modeled as a linear, shift-invariant system (*H(z)*) in feedback with a unilateral nonlinearity. Conditions for the absence of oscillations in such a system are then derived. The derivation follows a method originally presented by Mitra in which the existence of periodic oscillations is first assumed, then conditions leading to a contradiction are found. This approach is particularly attractive in that it exploits specific properties of the unilateral nonlinearity. The results developed here are presented graphically in the Nyquist plane, allowing direct comparison to other well-known criteria, such as Tsypkin's Condition. It is shown that the new criterion is much less conservative than Tsypkin's Condition.

1. INTRODUCTION

Unilateral constraints occur whenever bodies collide. A unilateral constraint allows all motions except those which would cause interpenetration of the bodies. Associated with any unilateral constraint is what we will term a "unilateral nonlinearity," which may be described as follows:



$$\xi(x) = \begin{cases} x & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \quad (1)$$

The subject of this paper is the effect that such a nonlinearity has on the stability of a physical system *simulation*. We are motivated particularly by the occurrence of limit cycles when real-time simulations (virtual environments) are coupled to a human operator via a haptic interface. It has frequently been observed that an operator can induce limit cycles simply by holding a haptic display against a virtual surface, without any voluntary attempt to stimulate oscillation (Minsky et al. 1990; Colgate et al., 1993). Such limit cycles present a safety hazard, and they destroy any illusion that the virtual surface is real. Limit cycles can usually be eliminated by reducing the stiffness of the virtual surface, but this also reduces the quality of the illusion.

For physical systems, the unilateral nonlinearity cannot be a source of instability or limit cycles because it cannot itself be a source of energy. In other words, two physical systems which are stable both in isolation and when rigidly coupled together, are stable when allowed to collide. The fact that collisions do not generate energy has been used to advantage, for instance, in modeling rigid body contact (Wang and Mason, 1992). For discrete time (or sampled-data) systems, however, it is possible that unilateral constraints will be directly responsible for instability. For example, consider a virtual surface implementation in which the haptic display position is measured at a fixed interval, and the computed reaction forces are held constant (by a zero-order hold) during each interval. Because the exact moment at which the virtual surface is released will be unknown, it is essentially certain that the reaction force will remain finite for some time after it should have returned to zero. Thus, the virtual surface will perform excess work on the operator, effectively becoming a source of energy.

It is unfortunately quite difficult to find strong stability conditions for nonlinear systems. One powerful tool is absolute stability theory, in which Lyapunov functions are found for linear time-invariant systems in feedback with static or time-varying nonlinearities (Narendra and Taylor, 1973). Existing results, such as Tsypkin's Criterion (1962), however, are too conservative to be of much use in the design of real-time simulations for haptic display. This is because such criteria are designed to account for significant amounts of uncertainty in the nonlinearity characteristic. The unilateral nonlinearity, in contrast, is quite well defined, a fact which should be exploited.

This paper presents a new and much less conservative stability condition for feedback systems incorporating a linear, shift invariant operator and a unilateral nonlinearity. The effect of a non-zero reference input is also included. This criterion, it will be seen, is not based on Lyapunov theory. It is, strictly speaking, a criterion for the absence of oscillations rather than a stability condition.

In the next section, the problem as it exists in haptic interface is described and a general problem statement is developed. In section 3, the new condition is derived. Section 4 presents a brief discussion.

2. PROBLEM STATEMENT

Figure 1 is a block diagram of a single-axis system for the haptic display of a "virtual wall." The system, as considered here, includes the human operator, and has been discussed in depth elsewhere (Colgate et al., 1993; Colgate and Schenkel, 1994; Colgate and Brown, 1994). It is desirable to find conditions on the linear part of the virtual wall simulation (*E(z)*) that guarantee system stability; however, this problem is confounded by the fact that the human operator dynamics are unknown. Because observed instabilities are typically of small-amplitude and well above the bandwidth of voluntary motion, it is reasonable to assume that the operator dynamics can be represented using a linear time-invariant (LTI) system. This system, however, is uncertain. A standard approach to accomodating uncertainty is simply to work with a **set** of LTI operators rather than a single operator. This approach will be taken here; however, the Nyquist plane criterion to be found will be as useful for a single operator as for a set.

In addition to generating forces in response to haptic display motions, the human operator must generate exogenous forces, or bias forces, to keep the haptic diplay against the virtual surface. Because observed limit cycles do not require any operator "pumping" or active excitation, and because they typically occur at frequencies above voluntary bandwidth, we will assume that the bias force is constant.

With these assumptions, it is straightforward to reduce the system in Figure 1 to that in Figure 2, although intersample behavior is lost in this reduction. This loss is not particularly troubling because observed oscillations always occur at frequencies well below the sampling frequency (sampling frequencies used in haptic display usually exceed the highest frequencies at which humans can even *detect* oscillation haptically).

Figure 2 describes the system that will be treated in this paper. The general assumptions that apply to this system are:

- *H(z)* is a stable, finite-dimensional, shift-invariant operator.
- $H(e^{j\omega T})$ lies within a known region of the complex plane at each frequency ω.

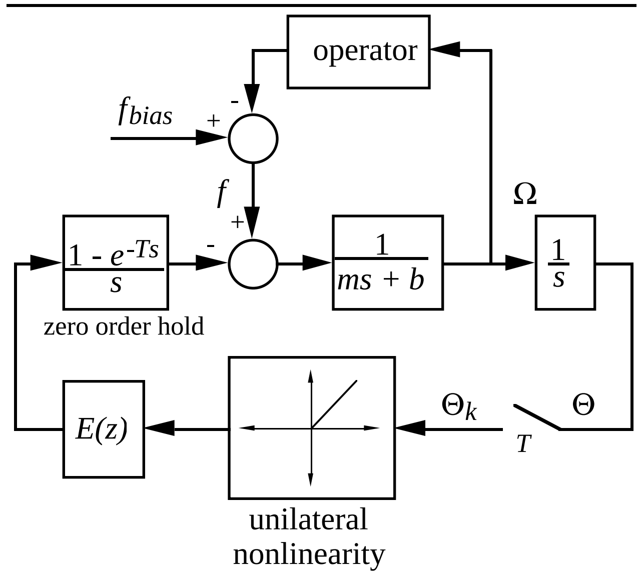


Figure 1. Simplified model of a haptic display system. Here, m is the inertia and b is the damping, while Θ is the position and Ω is the velocity of the haptic display hardware. f_{bias} is a bias force generated by the operator (f_{bias} is assumed to be constant), while f is the total force generated by the operator. T is the sampling period of the virtual environment simulation, and $E(z)$ is a pulse transfer function representing the behavior of the virtual wall.

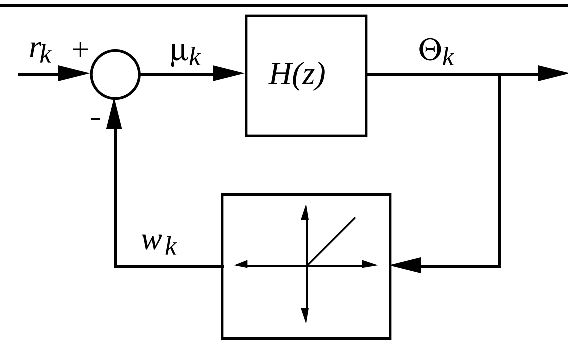


Figure 2. Feedback system nearly equivalent to that in Figure 1 with:

$$H(z) = E(z)\,(1 - z^{-1})\,Z\left\{\frac{1}{ms^3 + bs^2 + s^2 O(s)}\right\}$$

$$r_k = E^{-1}(1) f_{bias}$$

where Z represents the z-transform and O(s) the operator set. It is assumed that $E^{-1}(1)$ is finite (this is tantamount to assuming that the virtual wall has finite stiffness). The only difference from Figure 1 is in the bias signal. The two would be the same if we defined $r_k = E^{-1}(z) f_{bias}$; however, replacing z with 1 removes only the start-up effects (since f_{bias} is assumed constant) and does not change steady state behavior. In this way, r_k can be assumed constant.

- $(1 + H(z))^{-1}$ is stable. In other words, the closed loop system is stable in the absence of a unilateral constraint.
- $H(1) \geq 0$. In other words the d.c. gain of the composite system is positive.
- r_k is a positive constant, but otherwise unknown.

The problem is then to find conditions under which this system is guaranteed not to exhibit sustained oscillations (limit cycles).

3. CONDITIONS FOR THE ABSENCE OF OSCILLATIONS

One difficulty with analyzing the system in Figure 2 is handling the reference input. Stability analyses are usually performed for autonomous systems. Yet, the reference input plays an important role in this problem. For instance, a large positive reference may lead to strictly positive values of Θ_k, in which case the unilateral nonlinearity would be of no importance. This is akin to forcing a haptic display to remain inside a virtual wall by pressing very hard on it. A large negative reference would also render the unilateral nonlinearity unimportant. In the problem statement, therefore, we have left r_k unknown, though we have restricted it to positive values so that, in the case of a virtual wall, contact with the wall is certain to occur. It turns out, however, that Figure 2 is equivalent to an autonomous feedback system in which the unilateral nonlinearity is replaced with a set of sector-bounded nonlinearities.

To demonstrate this, r_k is first rewritten as:

$$r_k = \frac{H(z)}{1 + H(z)} r_k + \frac{1}{1 + H(z)} r_k \qquad (2)$$

Figure 2 is then manipulated as shown in Figure 3. Because we are interested in steady state solutions to this problem (i.e., limit cycles), and because r_k is assumed constant, it is clear that $H(z)(1 + H(z))^{-1} r_k$ will also be a constant, ρ_k:

$$\rho_k = \frac{H(1)}{1 + H(1)} r_k \qquad (3)$$

From Figure 3, the following relation is found:

$$y_k = -H(z)(\xi(y_k + \rho_k) - \rho_k) \qquad (4)$$

The expression $\xi(y_k + \rho_k) - \rho_k$ describes the shifted unilateral nonlinearity illustrated in Figure 4. Because ρ_k is a positive, unknown constant, the class of nonlinearities with which we are dealing is that illustrated in Figure 5. Thus, the original feedback system may be considered an autonomous feedback system in which the nonlinearity is known to fall within a particular class.

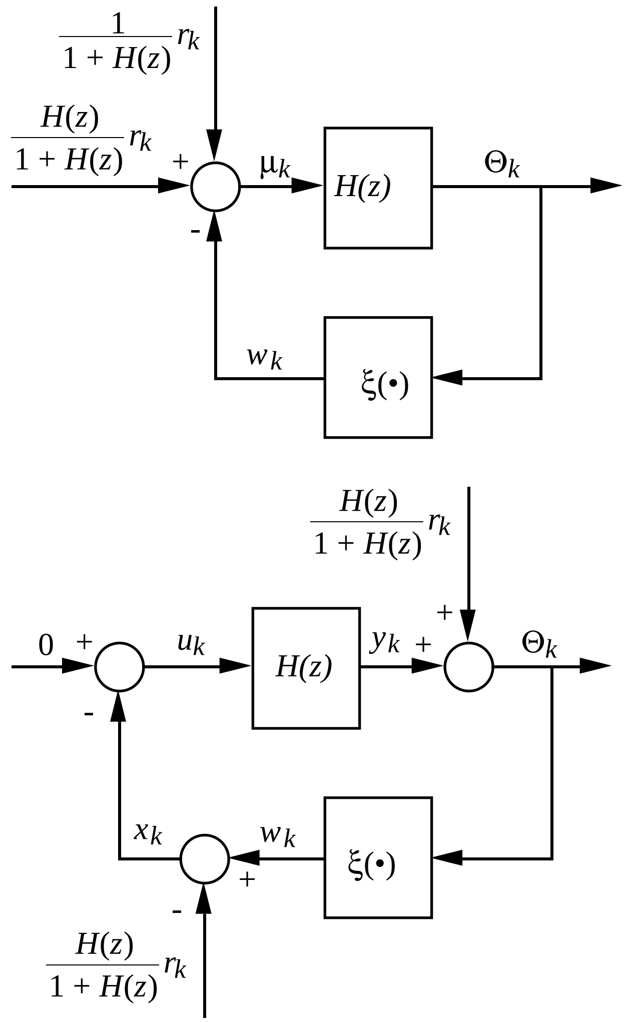


Figure 3. Block diagrams equivalent to that in Figure 2.

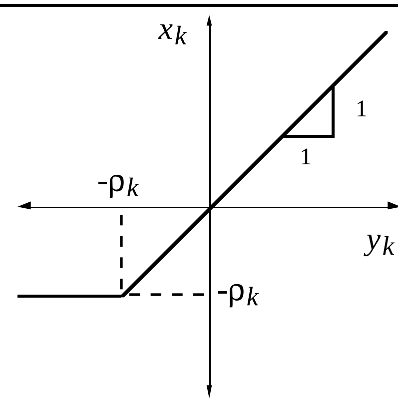


Figure 4. Plot of $\xi(y_k + \rho_k) - \rho_k$

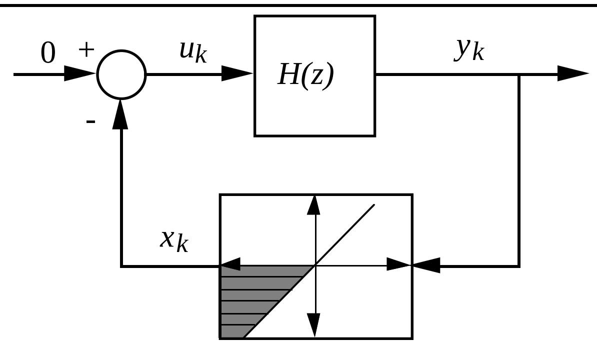


Figure 5. Autonomous feedback system used equivalent to Figure 2. The shaded region is that area of the plane covered by the shifted unilateral nonlinearity as ρ_k varies from 0 to ∞.

The problem as it is now posed can be recognized as a particular case of Lur'e and Postnikov's problem which considers feedback systems composed of a dynamic linear operator and a memoryless nonlinear operator, the latter known only to meet certain constraints, such as sector bounds (Zames, 1966). In this case, the sector bounds are associated with the shaded region in Figure 5. There are many well-known stability criteria for such systems, including the Circle (Zames, 1966), Popov (Narendra and Taylor, 1973), and Tsypkin (1962) criteria. Unfortunately, these are all too conservative to be useful for our problem, in large part because they assume too little about the nature of the nonlinearity. The class of nonlinearities shown above has associated with it certain properties which may be exploited to considerable advantage. Mitra (1977, 1978) has used such properties in deriving stability criteria for a similar system incorporating a saturation arithmetic nonlinearity. We will extend his analysis to the system in Figure 5. We will also go significantly beyond Mitra's development, which resulted in a rather arcane mathematical expression for stability, to present an intuitive graphical condition.

3.1 Properties of the Nonlinearity Set

Following Mitra (1977), we begin by identifying certain properties of the nonlinearity set. It is helpful to define e_k as:

$$e_k = x_k - y_k \geq 0 \tag{5}$$

Then, in terms of e_k and x_k, the unilateral constraint has the following properties:

$$e_k x_k \leq 0 \tag{6}$$

$$e_k(x_k - x_{k-q}) \leq 0 \quad \text{for every integer } q > 0. \tag{7}$$

The validity of these properties is easily proven (refer also to Figure 4). If $y_k \geq -\rho_k$, then $x_k = y_k$ and $e_k = 0$. In this case, both equations (5) and (6) hold. If $y_k \leq -\rho_k$, then $x_k = -\rho_k \geq y_k$, and $e_k \geq 0$. Also, for $y_k \leq -\rho_k$, $x_k = -\rho_k \leq x_{k-q}$ for every $q > 0$. In this case also, both inequalities (6) and (7) hold.

3.2 Description of *H(z)*

It will be convenient to express the pulse transfer function $H(z)$ in the following form:

$$H(z) = \frac{y_k}{u_k} = \frac{\sum_{j=1}^{m} a_j z^{-j}}{1 + \sum_{j=1}^{n} b_j z^{-j}} \tag{8}$$

The output y_k of $H(z)$ can be expressed in terms of the previous values of the input u_k and output y_k as well as the present value of the input u_k:

$$y_k = \sum_{j=0}^{m} a_j u_{k-j} - \sum_{j=1}^{n} b_j y_{k-j} = -\sum_{j=0}^{m} a_j x_{k-j} - \sum_{j=1}^{n} b_j y_{k-j} \tag{9}$$

3.3 Conditions for the Existence of Periodic Oscillations

Now assume that the system exhibits an oscillation with period N. Then we have the following equations:

$$\left\{ \begin{array}{c} x_1 = -a_0x_1 - a_1x_N - \ldots - a_mx_{N-m+1} - b_1y_N - b_2y_{N-1} - \ldots - b_ny_{N-n+1} + e_1 \\ x_2 = -a_0x_2 - a_1x_1 - \ldots - a_mx_{N-m+2} - b_1y_1 - b_2y_N - \ldots - b_ny_{N-n+2} + e_2 \\ \vdots \\ x_{N-1} = -a_0x_{N-1} - a_1x_{N-2} - \ldots - a_mx_{N-m-1} - b_1y_{N-2} - b_2y_{N-3} - \ldots - b_ny_{N-n-1} + e_{N-1} \\ x_N = -a_0x_N - a_1x_{N-1} - \ldots - a_mx_{N-m} - b_1y_{N-1} - b_2y_{N-2} - \ldots - b_ny_{N-n} + e_N \end{array} \right\} \tag{10}$$

The coefficients of x_k in this set of equations have a particular form known as *circulant* (see Appendix for a summary of circulant matrix properties). We may rewrite this set of equations in a matrix form:

$$\mathbf{M}_1\mathbf{X} + \mathbf{M}_2\mathbf{Y} = \mathbf{E}, \tag{11}$$

where

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{N-1} \\ e_N \end{bmatrix},$$

$$\mathbf{M}_1 = \begin{bmatrix} 1+a_0,\ 0,\ \ldots\ldots,\ 0,\ a_m,\ \ldots\ldots,\ a_5,\ a_4,\ a_3,\ a_2,\ a_1 \\ a_1,\ 1+a_0,\ 0,\ \ldots\ldots,\ 0,\ a_m,\ \ldots\ldots,\ a_5,\ a_4,\ a_3,\ a_2 \\ a_2,\ a_1,\ 1+a_0,\ 0,\ \ldots\ldots,\ 0,\ a_m,\ \ldots\ldots,\ a_5,\ a_4,\ a_3 \\ \vdots \\ 0,\ \ldots,\ 0,\ a_m,\ \ldots\ldots,\ a_5,\ a_4,\ a_3,\ a_2,\ a_1,\ 1+a_0,\ 0 \\ 0,\ \ldots\ldots,\ 0,\ a_m,\ \ldots\ldots,\ a_5,\ a_4,\ a_3,\ a_2,\ a_1,\ 1+a_0 \end{bmatrix},$$

and,

$$\mathbf{M}_2 = \begin{bmatrix} 0, \ldots\ldots, 0, b_n, \ldots\ldots, b_5, b_4, b_3, b_2, b_1 \\ b_1, 0, \ldots\ldots, 0, b_n, \ldots\ldots, b_5, b_4, b_3, b_2 \\ b_2, b_1, 0, \ldots\ldots, 0, b_n, \ldots\ldots, b_5, b_4, b_3 \\ \vdots \\ 0, .., 0, b_n, \ldots\ldots, b_5, b_4, b_3, b_2, b_1, 0, 0 \\ 0, \ldots, 0, b_n, \ldots\ldots, b_5, b_4, b_3, b_2, b_1, 0 \end{bmatrix}.$$

Since $\mathbf{M_1}$ and $\mathbf{M_2}$ are circulant matrices, they can be expressed as polynomials of the N×N primitive circulant matrix **P**:

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad (12)$$

$\mathbf{M_1}$ and $\mathbf{M_2}$ may be rewritten as polynomials of **P**:

$$\mathbf{M}_1 = \mathbf{I} + \sum_{j=0}^{m} a_j \mathbf{P}^j \qquad (13)$$

and

$$\mathbf{M}_2 = \sum_{j=1}^{n} b_j \mathbf{P}^j \qquad (14)$$

The properties (6) and (7) may be used together with (13) and (14) to derive quadratic equations which must be satisfied by a system supporting periodic oscillations:

$$\sum_{k=1}^{N} e_k x_k = \mathbf{X}'\mathbf{E} = \mathbf{X}'(\mathbf{M}_1\mathbf{X} + \mathbf{M}_2\mathbf{Y})$$

$$= \mathbf{X}'[\mathbf{M}_1 + \mathbf{M}_2(\mathbf{I} + \mathbf{M}_2)^{-1}(\mathbf{I} - \mathbf{M}_1)]\mathbf{X}$$

$$= \mathbf{X}'[\mathbf{I} + \sum_{j=0}^{m} a_j\mathbf{P}^j - (\sum_{j=1}^{n} b_j\mathbf{P}^j)(\mathbf{I} + \sum_{j=1}^{n} b_j\mathbf{P}^j)^{-1}(\sum_{j=0}^{m} a_j\mathbf{P}^j)]\mathbf{X}$$

$$= \mathbf{X}'[\mathbf{I} + (\sum_{j=0}^{m} a_j\mathbf{P}^j)(\mathbf{I} + \sum_{j=1}^{n} b_j\mathbf{P}^j)^{-1}]\mathbf{X} \qquad (15)$$

and

$$\sum_{k=1}^{N} e_k(x_k - x_{k-q}) = \mathbf{X}'(\mathbf{I} - \mathbf{P}^{-q})\mathbf{E} = \mathbf{X}'(\mathbf{I} - \mathbf{P}^{-q})(\mathbf{M}_1\mathbf{X} + \mathbf{M}_2\mathbf{Y})$$

$$= \mathbf{X}'[(\mathbf{I} - \mathbf{P}^{-q})\{\mathbf{M}_1 + \mathbf{M}_2(\mathbf{I} + \mathbf{M}_2)^{-1}(\mathbf{I} - \mathbf{M}_1)\}]\mathbf{X}$$

$$= \mathbf{X}'[(\mathbf{I} - \mathbf{P}^{-q})\{\mathbf{I} + \sum_{j=0}^{m} a_j\mathbf{P}^j - (\sum_{j=1}^{n} b_j\mathbf{P}^j)(\mathbf{I} + \sum_{j=1}^{n} b_j\mathbf{P}^j)^{-1}(\sum_{j=0}^{m} a_j\mathbf{P}^j)\}]\mathbf{X}$$

$$= \mathbf{X}'[(\mathbf{I} - \mathbf{P}^{-q})\{\mathbf{I} + (\sum_{j=0}^{m} a_j\mathbf{P}^j)(\mathbf{I} + \sum_{j=1}^{n} b_j\mathbf{P}^j)^{-1}\}]\mathbf{X} \qquad (16)$$

The last step in each of the above derivations follows from straightforward matrix manipulation and the commutability of circulant matrices (see Appendix). Each of these equations may be diagonalized by letting $\mathbf{Z} = \mathbf{V}^*\mathbf{X}$, where **V** is the unitary matrix of eigenvectors of the associated circulant matrix. The diagonalized equations may be rewritten as (the inequalities follow from (6) and (7)):

$$\sum_{k=1}^{N} |Z_k|^2 \mathrm{Re}\mu_k \le 0 \qquad (17)$$

and

$$\sum_{k=1}^{N} |Z_k|^2 \mathrm{Re}\lambda_k^q \le 0, \text{ for every } q \in \{1, 2, ..., N\} \qquad (18)$$

where μ_k and λ_k^q are eigenvalues of the circulant matrices. These eigenvalues take the following form:

$$\mu_k = 1 + (\sum_{j=0}^{m} a_j e^{-ij\theta})(1 + \sum_{j=1}^{n} b_j e^{-ij\theta})^{-1} = 1 + H(e^{i\theta}) \qquad (19)$$

and

$$\lambda_k^q = (1 - e^{iq\theta})\,\mu_k = (1 - e^{iq\theta})[1 + H(e^{i\theta})] \qquad (20)$$

where $i = \sqrt{-1}$ and $\theta = k2\pi/N$.

3.4 Mathematical Statement of the Stability Criterion

Sufficient conditions for stability may be found by contradiction. By requiring that the real part of either μ_k or λ_k^q be positive, equation (17) and/or equation (18) will not be satisfied, indicating that sustainted oscillations cannot exist:

$$\mathrm{Re}\,\mu_k > 0 \text{ or } \mathrm{Re}\{H(e^{i\theta})\} > -1, \qquad (21)$$

or

$$\mathrm{Re}\,\lambda_k^q > 0 \text{ or } \mathrm{Re}\{H(e^{i\theta})\} + \frac{\sin(q\theta)}{1 - \cos(q\theta)}\mathrm{Im}\{H(e^{i\theta})\} > -1 \qquad (22)$$

Only (21) or (22) for some value of q ($q \in \{1, 2, ..., N\}$) must be satisfied to ensure that oscillations cannot exist. One way to express this is simply to compute the maximum of the N+1 real parts, and require that this be positive at all values of θ between 0 and π. Another way is to require that there exist non-negative α and β_q such that:

$$\alpha \mathrm{Re}\mu_k + \sum_{q=1}^{N} \beta_q \mathrm{Re}\lambda_k^q > 0 \text{ for all } \theta \text{ in } [0,\pi], \text{ where } \theta = k2\pi/N \qquad (23)$$

Either approach leads to a somewhat arcane interpretation of (21) and (22). Much more insight is gained from a graphical interpretation.

3.5 Graphical Interpretation

(21) and (22) may be considered inequality contraints in the Nyquist Plane of *H(z)*. Figure 6 is an illustration of these inequalities at a particular value of θ ($\theta = \pi/4$). The smallest value of N that can produce $\theta = \pi/4$ is N = 8. However, if we choose k>1, we may choose other, larger values of N. Thus, q may be chosen as *any* integer, one or greater. The reader may wonder, then, why we have shown only the values q = 1, 2, ..., 7 in this figure. The reason is as follows. For q = 8, equation (22) is indeterminant, and must be rewritten as (this is actually the form from which (22) was derived):

$$[1 - \cos(q\theta)]\, \mathrm{Re}\{H(e^{j\theta})\} + \sin(q\theta)\, \mathrm{Im}\{H(e^{j\theta})\} > -[1 - \cos(q\theta)] \tag{24}$$

Because $q\theta = 2\pi$, this equation reduces to the impossible condition 0 > 0; thus, this relation will not be satisfied, and we must go on to other values of q. Higher values of q, however, simply lead to repeated conditions.

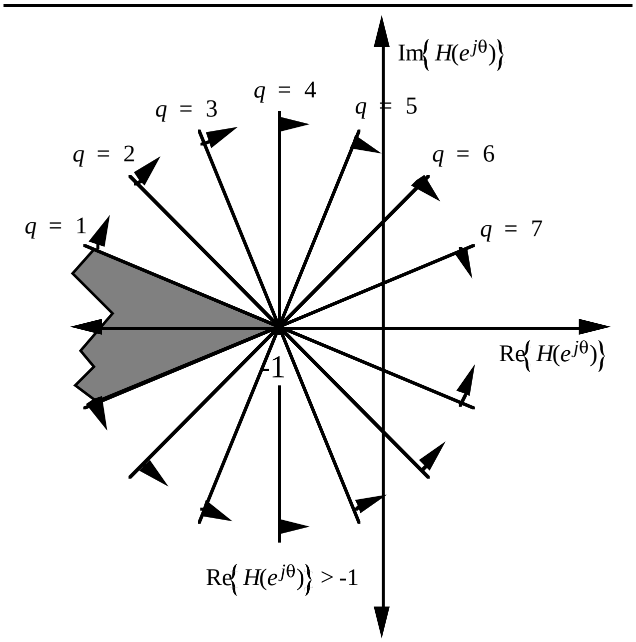


Figure 6. Equations (21) and (22) shown in the Nyquist plane of *H(z)* for $\theta = \pi/4$. Each inequality divides the plane into two halves. The arrows point to the half that satisfies the inequality. The vertical line is both (21) and (22) for q = 4. The shaded region is the only part of the plane which does not satisfy at least one inequality.

It is clear in Figure 6 that most of the inequality constraints play no role. This will be true in general. It is clear from graphical construction that, if $\theta = k2\pi/N$, the two values of q that lead to the most restrictive inequalities are q = 1 and q = N-1; moreover, the latter produces an inequality equivalent to that produced by q = -1. Thus, a final, compact statement of the stability criterion is:

$$m(\theta) = \frac{\sin(\theta)}{1 - \cos(\theta)}$$

$$\max\Big(\mathrm{Re}\{H(e^{j\theta})\} + m(\theta)\, \mathrm{Im}\{H(e^{j\theta})\},\ \mathrm{Re}\{H(e^{j\theta})\} - m(\theta)\, \mathrm{Im}\{H(e^{j\theta})\}\Big) > -1 \quad \text{for all } \theta,\ 0 \le \theta \le \pi \tag{25}$$

This result is similar to the Popov Criterion in that it defines a frequency-dependent region of the plane which must be avoided. Instead of the half-plane, however, the forbidden region is a wedge. The cone angle of the wedge is 0° at zero frequency and 180° at the Nyquist frequency.

4. CONCLUSIONS

A new stability condition for discrete time feedback systems incorporating a unilateral constraint has been presented. This condition is easily interpreted in the Nyquist plane, and therefore can be compared to published absolute stability criteria which may also be interpreted in the Nyquist plane. Among these, perhaps the most widely applied to discrete time systems is Tsypkin's criterion, which is simply $\mathrm{Re}\{H(e^{j\theta})\} > -1$. It is clear that the present condition is much less conservative. The new criterion may be usefully applied to the stability analysis of haptic displays, though to do so will require a measure of the opeator set, $O(s)$. In addition, the effects of the operator set, the haptic display hardware, and the sample-and-hold must be "backed out" of *H(z)* by straightforward manipulation in order to put constraints on the virtual evironment behavior *E(z)*,.

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APPENDIX

Let **P** denote the primitive N×N circulant matrix:

$$\mathbf{P} = \begin{bmatrix} 0, 0, 0, 0, \ldots\ldots, 0, 0, 0, 0, 1 \\ 1, 0, 0, 0, \ldots\ldots, 0, 0, 0, 0, 0 \\ 0, 1, 0, 0, \ldots\ldots, 0, 0, 0, 0, 0 \\ 0, 0, 1, 0, \ldots\ldots, 0, 0, 0, 0, 0 \\ 0, 0, 0, 1, \ldots\ldots, 0, 0, 0, 0, 0 \\ \vdots \\ 0, 0, 0, 0, \ldots\ldots, 0, 0, 1, 0, 0 \\ 0, 0, 0, 0, \ldots\ldots, 0, 0, 0, 1, 0 \end{bmatrix}. \qquad \text{(A1)}$$

Then,

$$\mathbf{P}^2 = \begin{bmatrix} 0, 0, 0, 0, \ldots\ldots, 0, 0, 0, 1, 0 \\ 0, 0, 0, 0, \ldots\ldots, 0, 0, 0, 0, 1 \\ 1, 0, 0, 0, \ldots\ldots, 0, 0, 0, 0, 0 \\ 0, 1, 0, 0, \ldots\ldots, 0, 0, 0, 0, 0 \\ 0, 0, 1, 0, \ldots\ldots, 0, 0, 0, 0, 0 \\ 0, 0, 0, 1, \ldots\ldots, 0, 0, 0, 0, 0 \\ \vdots \\ 0, 0, 0, 0, \ldots\ldots, 0, 1, 0, 0, 0 \\ 0, 0, 0, 0, \ldots\ldots, 0, 0, 1, 0, 0 \end{bmatrix}, \qquad \text{(A2)}$$

All higher powers of P are found similarly, simply by “circulating” the rows or columns. Properties which follow and others that are related are:

$$\mathbf{P}^N = \mathbf{P}^0 = \mathbf{I} \qquad \text{(A3)}$$

$$\mathbf{P}^{-1} = \mathbf{P}^{N-1} = \mathbf{P}' \qquad \text{(A4)}$$

$$\mathbf{P}^{-2} = \mathbf{P}^{N-2} = [\mathbf{P}^2]' \qquad \text{(A5)}$$

etc., where ' denotes the transpose.

Any N×N circulant matrix can be expressed as a polynomial of the primary N×N circulant matrix **P**, for example:

$$\mathbf{A} = \begin{bmatrix} a_1, a_2, a_3, \ldots\ldots, a_{m-1}, a_m \\ a_m, a_1, a_2, \ldots\ldots, a_{m-2}, a_{m-1} \\ a_{m-1}, a_m, a_1, \ldots\ldots, a_{m-3}, a_{m-2} \\ \vdots \\ a_3, a_4, a_5, \ldots\ldots, a_1, a_2 \\ a_2, a_3, a_4, \ldots\ldots, a_m, a_1 \end{bmatrix}$$

$$= a_1P^0 + a_mP^1 + a_{m-1}P^2 + \ldots + a_2P^{m-1} = \sum_{j=1}^{m} a_jP^{1-j+m} \qquad \text{(A6)}$$

The k^{th} eigenvalue of the primary N×N circulant matrix **P** is:

$$\lambda_k = e^{-ik2\pi/N}, \qquad \text{(A7)}$$

where k = 1, 2, ..., N, and $i = \sqrt{-1}$. It is easy to prove in addition that if λ_k is the eigenvalue **P**, then λ_k^j is the eigenvalue of $\mathbf{P}^j$ for j = 1, 2, ..., N. It is also easy to prove that if $\mathbf{v}_k$ is the eigenvector of **P**, then $\mathbf{v}_k$ is also the eigenvector of $\mathbf{P}^j$. With these results in hand, it is readily shown that the eigenvalue of circulant matrix $\mathbf{C} = \sum_{j=1}^{N} c_j\mathbf{P}^j$ is $\sum_{j=1}^{N} c_j\lambda_k^j$, or $\sum_{j=1}^{N} c_je^{-ijk2\pi/N}$ (where k = 1, 2, ..., N) associated with the eigenvector $\mathbf{v}_k$.

All circulant matrices of a given dimension share the same eigenvectors, which may be expressed as:

$$\mathbf{v}_k = a[e^{ik2\pi/N}, e^{i2k2\pi/N}, e^{i3k2\pi/N}, \ldots, e^{iNk2\pi/N}]'$$

$$= a[e^{ik2\pi/N}, e^{i2k2\pi/N}, e^{i3k2\pi/N}, \ldots, 1]'$$

$$= a[e^{-ik2\pi/N}, e^{-i2k2\pi/N}, e^{-i3k2\pi/N}, \ldots, 1]^* \qquad \text{(A8)}$$

where a is any arbitrary constant, ' denotes the transpose, and * denotes the conjugate transpose. If we choose $a = 1/\sqrt{N}$, the eigenvectors form an orthonormal set. If the eigenvectors are so scaled, and arranged as the columns of a square matrix **V**, then this matrix is unitary. If **C** is a circulant matrix, then:

$$\mathbf{V}^{-1}\mathbf{C}\mathbf{V} = \mathbf{V}^*\mathbf{C}\mathbf{V} = \mathbf{dg}\,(\lambda_1, \lambda_2, \ldots, \lambda_N) \qquad \text{(A9)}$$

where λ_k are the eigenvalues of **C**.

Now suppose **X** is an N×1 real column vector and we want to evaluate **X’CX**. One way is to diagonalize **C**. Let

$$\mathbf{Z} = \mathbf{V}^*\mathbf{X}, \qquad \text{(A10)}$$

Note that **Z** is an N×1 column vector and can be expressed as

$$\mathbf{Z} = [Z_1, Z_2, \ldots, Z_N]', \qquad \text{(A11)}$$

where $Z_1, Z_2, \ldots, Z_N$ can be complex numbers. Consequently,

$$
\begin{aligned}
\mathbf{X}'\mathbf{C}\mathbf{X} &= \mathbf{Z}^{*}\mathbf{V}^{*}\mathbf{C}\mathbf{V}\mathbf{Z} \\
&= \mathbf{Z}^{*}(\mathbf{V}^{*}\mathbf{C}\mathbf{V})\mathbf{Z} \\
&= \mathbf{Z}^{*}\mathbf{dg}\,(\lambda_1, \lambda_2, \ldots, \lambda_N)\mathbf{Z} \\
&= \sum_{k=1}^{N} |Z_k|^2 \mathrm{Re}\,\lambda_k
\end{aligned}
\tag{A12}
$$

A final point is that, because all circulant matrices of a given dimension have the same eigenstructure, these matrices are commutative.