Chapter 2

Passivity

The concept of a passive physical system will play an important role in the sequel. The purposes of this chapter are to provide the reader insight into passivity, and to derive the mathematical statements of passivity that will prove useful in subsequent analyses.

2.1 Thermodynamic Concepts

Passivity is closely related to the thermodynamic concept of availability, due to Keenan [15,48]. Availability may be understood as follows.

Consider (following [15]) a system immersed in an atmosphere as shown in Figure 2.1. We are interested in determining the amount of useful work that can be extracted from this system as it undergoes a change in state (from state 1 to state 2). The first law of thermodynamics requires the following:

\[(Q_{1-2})_{\text{rev}} - (W_{1-2})_{\text{rev}} = E_2 - E_1\]  \hspace{1cm} (2.1)

where \((Q_{1-2})_{\text{rev}}\) is the heat delivered to the system by a reversible process from state 1 to state 2; \((W_{1-2})_{\text{rev}}\) is the work done by the system; and \(E\) is the energy of the system. The second law of thermodynamics may be invoked to rewrite this equation.
for a general, possibly irreversible, process:

$$W_{1-2} \leq \int_{1}^{2} T \, dS - (E_2 - E_1)$$  \hspace{1cm} (2.2)$$

where $T$ is the temperature and $S$ is the entropy of the system. Not all of $W_{1-2}$ will be useful work, however. Some portion of the total work transfer will be required to displace the atmosphere:

$$(W_{1-2})_{useful} \leq \int_{1}^{2} T \, dS - \int_{1}^{2} P_{atm} \, dV - (E_2 - E_1)$$  \hspace{1cm} (2.3)$$

where $V$ is the volume of the system. The integrals on the right side of this equation will be maximized for a reversible process which, because the atmosphere is at constant temperature and pressure, will be isothermal and isobaric. Thus:

$$(W_{1-2})_{useful} \leq -[(E + P_{atm}V - T_{atm}S)_2 - (E + P_{atm}V - T_{atm}S)_1]$$  \hspace{1cm} (2.4)$$

We now define the property $\Phi = E + P_{atm}V - T_{atm}S$, and write:

$$(W_{1-2})_{useful} \leq - (\Phi_2 - \Phi_1)$$  \hspace{1cm} (2.5)$$
The quantity $\Phi$ has a minimum value, corresponding to the system state $P = P_{atm}$ and $T = T_{atm}$. Thus, Keenan has defined the availability, $\Lambda$, as:

$$\Lambda = \Phi - \Phi_{min}$$  \hfill (2.6)

Equation 2.5 can be rewritten in terms of the availability as follows:

$$(W_{1-2})_{useful} \leq \Lambda_1 - \Lambda_2$$  \hfill (2.7)

Now that we have an expression for the amount of useful work that can be extracted from a system, we are prepared to consider passivity. However, because the first and second laws of thermodynamics guarantee the existence of such a $\Phi_{min}$, thermodynamics texts generally do not forge a link between availability and passivity. In order to do this, it is necessary to postulate the existence of systems for which $\Phi$ has no lower bound. For such systems, the availability ($\Lambda$) would be infinite. Because these systems could supply infinite useful work, they would clearly not be passive in any intuitive sense. Conversely, those systems for which $\Phi$ is bounded from below ($\Phi_{min}$ exists) could supply only finite useful work, and should be termed "passive". This is, in fact, the concept of passivity which has received the most widespread acceptance in the engineering literature [90,93].

But clearly something is amiss. Any physical system which can be described as shown in Figure 2.1, and which has no lower bound on $\Phi$, is nothing less than a perpetual motion machine of the second kind! Is it true then, that the laws of thermodynamics prohibit the existence of active systems? The answer, which is surely obvious when considered on a universal scale, is yes. Thus, we have arrived at our first statement of passivity, which follows from the postulate that all physical systems must obey the laws of thermodynamics:

**Passivity 1 (Thermodynamic).** All physical systems are passive.
Of course, this statement poses a practical problem because engineers have access to a plethora of devices, such as electric motors and amplifiers, which are commonly referred to as active. The problem is that, when speaking of active (as opposed to passive) systems, engineers typically ignore interaction with certain other systems, not shown in Figure 2.1. For instance, it is inadequate from a thermodynamic point of view to model a motor as interacting with only its load and surroundings; yet, this is an approach which is frequently and justifiably taken in the process of modeling mechanical devices. The interaction with the electrical power source should also be included, and even this source should not be considered infinite if the laws of thermodynamics are to be satisfied in full.

Such a modeling effort would, however, while expanding the realm of validity, do little to improve the utility of the result. The point is that engineers are generally forced to generate models based upon limited\(^1\) sets of observations, and to describe these models with mathematical equations which have limited bounds of validity. As a consequence, models often predict that certain systems are capable of performing infinite work on their environments, and extrapolation based on limited observations often bears this out. Within this context, the distinction between active and passive systems becomes useful. The next section explores the origin of active models.

### 2.2 Active Models

This section is intended to bridge the gap between the thermodynamic concepts of the last section, which do not allow active systems at all, and the concepts of the next section, which make a mathematical distinction between passivity and activity. We will make this bridge by examining the specific mechanisms with which mathematical mod-

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\(^{1}\)For instance, engineers do not have infinite time for observation.
models can predict active behavior. For the purposes of illustration, these mechanisms will be divided into three categories, although these categories are not rigorously distinct.

The first and probably most ubiquitous mechanism is simply the use of sources in modeling. A battery, modeled as a voltage source, is an example. Wyatt, et al. [93] have pointed out that this sort of active model is due to multiple time scales. The time scale of interest for the electric circuit of which the battery is a part is generally too short for the battery to discharge significantly, thus the battery is modeled as not discharging at all. This type of extrapolation tends to be easily justified, because the model matches the relevant observation. Sources will be standard modeling elements for the rest of this document.

The second mechanism is the use of models with limited bounds of validity (though not necessarily limited in time), as mentioned in the previous section. To elaborate, consider a charged \((-q\) particle located midway between two fixed spheres, each of charge \(+q\) (Figure 2.2).

The force on this particle is zero, but it is clearly in a state of (locally) unstable equilibrium. If we were to linearize the restoring force as a function of displacement about the center position, the result would appear to be the constitutive equation of a negative spring:

\[
\delta F = - \left( \frac{4kq^2}{l^5} \right) \delta x
\]

where \(k = 8.9879 \times 10^9 \text{ Nm}^2\text{C}^{-2}\). The potential energy of this linearized model (relative
to the energy as $\delta x \to \infty$) is infinite for any finite $\delta x$, suggesting that the system can supply infinite energy. However, analysis of the nonlinear model based upon Coulomb's Law, together with the assumptions that the spheres behave as point charges and cannot penetrate one another, leads to the following maximum energy which can be extracted from the system:

$$E = kq^2 \left( \frac{1}{R+r} + \frac{1}{2l-R-r} - \frac{2}{l} \right)$$

For $r > 0$ and $R > 0$, this solution is certainly not infinite. Of course, not even the nonlinear model is completely valid, but the point is that active behavior can be a consequence of a particular model's limitations.

The final mechanism, which presumes the first (sources), is somewhat more subtle. It arises because mathematical representations, such as state equations, of physical systems are quite capable of disguising energy sources. This can occur when a particular source is state-dependent; i.e., when a system is feedback controlled.

Consider, for instance, the system in Figure 2.3. If we were to implement the
following feedback law,
\[
\begin{bmatrix}
F_x \\
F_y
\end{bmatrix} =
\begin{bmatrix}
0 & -K \\
K & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

the result would be a system described by the following state equations:
\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\ddot{x} \\
\ddot{y}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-K & -K & 0 & 0 \\
-K & -K & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
\dot{x} \\
\dot{y}
\end{bmatrix}
\]

Although no energy source is evident in these equations, the vector field relating steady state forces to displacements has non-zero curl. This suggests that infinite energy can be extracted from this closed-loop system, simply by allowing the mass to move continually in a closed, counterclockwise path (while turning a crank, for instance).

Note that feedback does not necessarily result in active behavior. We could just as well have chosen the following feedback law,
\[
\begin{bmatrix}
F_x \\
F_y
\end{bmatrix} =
\begin{bmatrix}
0 & -K \\
-K & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

which would have resulted in closed-loop behavior indistinguishable from that of an appropriately designed collection of springs and masses.

The result, however, is that state-dependent sources are an important mechanism for generating active behavior, but the presence of these sources is not necessarily evident on the basis of a state description alone. Nevertheless, state equations rather than physical models provide the starting point for many of the theoretical developments found in the network analysis and control systems literature, so that an analysis of passivity based upon a state equation formulation proves quite useful.

### 2.3 General State Space Concepts of Passivity

In this section the results of Willems [90] and Wyatt, et al. [93] concerning passivity will be informally presented. This presentation is intended to provide the reader with
an introduction to the state-space theory of passivity and to pave the way for the linear analyses of the next section without fostering the naive belief that linear concepts of passivity have global application. It is not the object of this presentation to provide a rigorous statement of passivity. Rigor, as well as a number of curious examples that justify the need for rigor, can be found in [93].

The concepts presented in [90,93]$^2$ apply to finite-dimensional, time-invariant systems described by state equations:

\begin{align}
\dot{x} &= g(x,u) \\
y &= h(x,u)
\end{align}

(2.8)

The system described by these equations is taken to have $n$ interaction ports. A power input function, $p(x,u)$, which is the sum of the products of generalized effort and flow at each of the ports, is defined:

\[ p(x,u) = \langle e(x,u), f(x,u) \rangle \]

(2.9)

Before making a definition of passivity, several assumptions are necessary:

1. $g$, $h$, $e$, and $f$ are continuous functions.

2. For every initial condition $x_0$ and input $u(t)$, there exists a unique solution $x(t)$ to the differential equation $\dot{x} = g(x,u)$.

3. The set of inputs is translation invariant and closed under concatenation; i.e., if $u(t)$ is an admissible input, so is $u(t + \tau)$, and if $u_1(t)$, $t_1 \leq t \leq t_2$, and $u_2(t)$, $t_2 \leq t \leq t_3$, are admissible inputs, so is $u_1$ followed by $u_2$.

4. $p(x,u)$ is locally $L^1$, i.e.:

\[ \int_{t_1}^{t_2} \| p(x,u) \| \, dt < +\infty \text{ for every choice of } t_1 \geq 0, t_2 \geq 0. \]

$^2$The analyses of passivity found in Willems and Wyatt, et al. are quite similar—identical for our purposes. The notation used here will be that of Wyatt, et al.
Given these assumptions, the \textit{available energy} (Wyatt, et al.) or \textit{available storage} (Willems) can be defined and related to passivity. The available energy is defined as follows:

\[ E_A(x) \triangleq \sup_{x \in \mathbb{R}^n} \left\{ - \int_0^T p(x, u) \, dt \right\} \quad (2.10) \]

where the notation \( x \to \) indicates that the supremum is taken over all state trajectories starting in the state \( x \) at \( t = 0 \), and where the supremum is taken over all admissible inputs. Note that \( E_A(x) \) is a non-negative function since it is the supremum over a set of numbers which include 0 \( (T = 0) \). The intuitive relation of the available energy to availability should be fairly evident—both represent the maximum work that can be extracted from a particular system. Specifically, the available energy for a particular initial state \( x \) represents the maximum energy (all of which may be considered to be useful work) which can be extracted from the system when its initial state is \( x \).

At this point a statement of passivity (formally, a theorem) is straightforward:

\textbf{Passivity 2a (State-Space).} A dynamic system is passive if, for each initial state \( x \),

\[ E_A(x) < +\infty. \]

Otherwise, it is active.

This is clearly a rather weak definition of passivity, particularly in that \( E_A \) must be calculated for every initial state \( x \). We can alleviate this problem somewhat by considering only those systems which are \textit{completely controllable}. For a completely controllable system there always exists a control \( u(t), 0 \leq t \leq T < \infty \), which will take a system from each state \( x_0 \) to any other state \( x_1 \). Further, because of assumption (4), this transfer will consume or produce only finite energy, and, as a consequence, we need examine only one initial state:

\textbf{Passivity 2b.} A system with a completely controllable state representation is passive iff \( E_A(x_0) < +\infty \), where \( x_0 \) is any initial state.

Finally, Wyatt et al. have strengthened the definition further for those systems
which have states of zero stored energy, called *relaxed states*. A state \( x \) is said to be relaxed if \( E_A(x) = 0 \).

**Passivity 2c.** If a system with a completely controllable state representation has a relaxed state \( x^* \), then the system is passive.

An alternative statement of passivity is in terms of internal energy functions or storage functions. \( E_I(x) \) is an internal energy function of a system if:

\[
E_I(x(t_2)) - E_I(x(t_1)) \leq \int_{t_1}^{t_2} p(x(t), u(t)) \, dt
\]  

(2.11)

for all inputs \( u(t) \), and all \( 0 \leq t_1 \leq t_2 \). An internal energy function is a function of the state which is bounded from below and increases along trajectories in the state space more slowly than the rate at which energy is delivered to the ports. Note that, if a system is passive, \( E_A \) is an internal energy function.

The following theorem relates internal energy to passivity:

**Passivity 3 (State-Space).** Given a system with a state representation subject to the same assumptions used earlier, the system is passive iff there exists an internal energy function \( E_I(x) \) defined over the state space.

If the internal energy function is sufficiently smooth, then a differential version of equation 2.11 exists:

\[
\dot{E}_I(x, u) \leq p(x, u)
\]

(2.12)

The use of an internal energy function can be attractive because of its potential application as a Lyapunov function. It is important to realize, however, that an internal energy function need not qualify as a Lyapunov function, as the latter is subject to more stringent requirements on shape [86].

This completes our brief tour of nonlinear systems passivity. The next section will treat the more specific class of linear, time-invariant, finite-dimensional systems.
2.4 Passivity of Linear Systems

The purpose of this section is to derive quantitative statements of the passivity of linear, time-invariant, finite-dimensional systems described as follows:

\[
\begin{align*}
\dot{x} &= A_{k \times k} x + L_{k \times n} u \\
y &= C_{n \times k} x + D_{n \times n} u
\end{align*}
\]  

(2.13)

where \( u \) and \( y \) are a hybrid pair [4] of power duals, i.e., generalized efforts and flows associated with the \( n \) interaction ports (Figure 2.4). The developments in this section are fairly standard, similar to those found in several texts on network analysis. In particular, this presentation draws heavily upon the analyses of Kuh and Rohrer [53], Anderson and Vongpanitlerd [4], and the seminal work of Brune [13].

We will assume that the system above is a minimal realization [4,91], which is to say that it is both controllable and observable. The assumption of controllability was motivated in the previous section; with it, the initial state is not a consideration. The additional assumption of observability allows us to tie together the external (in-
put/output) and internal (state-based) analyses of passivity. In particular, it enables us to disregard those pathological cases in which the Lyapunov energy—defined to be a positive definite quadratic function of state\(^3\)—grows without bound due to an unstable mode which also happens to be unobservable.

Certain assumptions must also be made regarding the inputs:

1. The set of inputs is closed under concatenation (see page 43).

2. The set of inputs is translation invariant (see page 43).

3. The set of inputs is \(L^2(-\infty, t]\), i.e.:

\[
\int_{-\infty}^{t} \| u(t) \|^2 \, dt < +\infty
\]

The third assumption is essentially a boundedness criterion on the input; the utility of each of these assumptions will become evident in the sequel. It should be noted that none of the assumptions is restrictive for systems of engineering interest.

The statement of passivity in terms of internal energy (equation 2.11) can now be applied to linear \(n\)-ports.

A linear \(n\)-port is passive iff:

\[
\int_{t_1}^{t_2} u'(\tau)y(\tau) \, d\tau \geq E(t_2) - E(t_1)
\]  

(2.14)

for all admissible \(n \times 1\) \(u(t)\) and \(y(t)\) and all \(t_1\) and \(t_2 > t_1\). \(E(t)\) is now taken to be the Lyapunov energy, which, for a minimal realization, is a suitable internal energy function. The Lyapunov energy is bounded below by zero. The derivative form of

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\(^3\)The Lyapunov energy is defined as follows: \(E_{\text{Lyapunov}} = x'Px\), where \(P\) is a positive definite matrix. It is used here rather than the internal energy to emphasize the dependence on all states of the system. The internal energy functions defined in the previous section can ignore an unstable, unobservable mode, Lyapunov energy cannot.
Equation 2.14 is also a suitable statement of passivity:

\[ u'(t)y(t) \geq \frac{dE}{dt} \]  

(2.15)

Equations 2.14 and 2.15, despite their intuitive appeal, can be somewhat unwieldy because they require measurement of the port variables \( u \) and \( y \) as well as the energy. A criterion which does not require the measurement of energy can be found as follows. Because the linear system is controllable, there exists some control which will take the system from its state at \( t_2 \) to the state \( x = 0 \) at \( t_2 < t_3 < +\infty \). Because the inputs are closed under concatenation, the following equation holds:

\[ \int_{t_1}^{t_3} u'(\tau)y(\tau) \, d\tau \geq -E(t_1) \]  

(2.16)

This equation essentially states that a passive system can deliver no more energy than that amount which is initially stored in it. Next, we make the assumption that the Lyapunov energy approaches zero (i.e., the system is at rest) as \( t_1 \to -\infty \), thus:

**Passivity 4 (Linear n-Port).** A linear \( n \)-port is passive iff:

\[ \int_{-\infty}^{t} u'(\tau)y(\tau) \, d\tau \geq 0 \]  

(2.17)

This is the statement of passivity which will be used for most of what follows, which is the development of practical passivity criteria in terms of both input/output and state-space descriptions. The input/output criteria will be considered first, but for historical reasons only (they predate the state-space criteria by approximately 30 years); they are in no way the more fundamental of the two.

### 2.4.1 Input/Output Passivity Criteria

The various input/output passivity criteria will be generated in the frequency domain. When working in the frequency domain, it can be useful to express physical quantities
of interest as the real parts of complex quantities. Thus, we will consider power duals $u$ and $y$ of the following form:

\[
\begin{align*}
    u(t) &= u_0 e^{s t} \cos(\omega t) = Re\{u_0 e^{s t}\} \\
y(t) &= y_0 e^{s t} \cos(\omega t + \phi) = Re\{y_0 e^{s t + j\phi}\}
\end{align*}
\] (2.18)

where $s = \sigma + j\omega$. Here, the the vectors $u_0$ and $y_0$ are real, and the assumption is made (temporarily) that the components of $y$ all share the same phase. The power delivered to this $n$-port at any time $t$ is:

\[
u'(t)y(t) = u'_0 y_0 e^{2\sigma t} \cos(\omega t) \cos(\omega t + \phi)
\] (2.19)

With the aid of the following trigonometric identity, which identifies the non-periodic part of equation 2.19, the net energy delivered at any time $t$ may be evaluated:

\[
\cos(\omega t) \cos(\omega t + \phi) = \frac{1}{2} \cos(\phi) + \frac{1}{2} \cos(2\omega t) \cos(\phi) - \frac{1}{2} \sin(2\omega t) \sin(\phi)
\]

Thus,

\[
\int_{-\infty}^{t} u'(r)y(r) \, dr = \frac{1}{4\sigma} u'_0 y_0 e^{2\sigma t} \cos(\phi) \] (2.20)

for $\sigma \geq 0$.

Now consider an alternative definition of passivity for linear systems:

A linear $n$-port is passive iff:

\[
Re \left\{ \int_{-\infty}^{t} u^H(r)y(r) \, dr \right\} \geq 0
\] (2.21)

for all admissable $u(t)$ and $y(t)$ and all $t$. In this case, $u$ and $y$ may be complex quantities, not just the real parts. Given $u$ and $y$ in the form of equations 2.19, definition 2.21 reduces to:

\[
Re \left\{ \int_{-\infty}^{t} u'_0 e^{s r} y_0 e^{s r + j\phi} \, dr \right\} = \frac{1}{2\sigma} u'_0 y_0 e^{2\sigma t} \cos(\phi) \geq 0
\] (2.22)
where \( s^* \) is the complex conjugate of \( s \), and \( \sigma \geq 0 \).

Comparison with equation 2.20, reveals that the expression in equation 2.22 is simply twice the net energy delivered to the system. This result is readily generalized to the case in which \( u_0 \) and \( y_0 \) are complex vectors, and to the case in which each component of the vectors \( u \) and \( y \) may have a distinct phase. The consequence is that equation 2.21 is an acceptable statement of passivity, providing results identical to equation 2.17, and further, offering simpler computation.

The strength of a complex-variable approach is that it lets us examine passivity in terms of the properties of the driving point impedance matrix\(^4\) relating the Laplace transforms of \( u \) and \( y \):

\[
y(s) = Z(s)u(s) \tag{2.23}
\]

where \( Z(s) = D + C(sI - A)^{-1}L \)

It is now necessary to consider carefully the relation of the class of inputs \( u(t) \) to the inputs \( u(s) \). The assumptions that \( u(t) \) is translation invariant and \( L^2(-\infty, t] \) are now important. Because \( u(t) \) is translation invariant it is clear that the following equation holds:

\[
\int_{-\infty}^{t} \| u(\tau - t) \|^2 \, d\tau < +\infty
\]

Introducing the change of variables \( \tau^* = \tau - t \):

\[
\int_{-\infty}^{0} \| u(\tau^*) \|^2 \, d\tau^* < +\infty
\]

Therefore the class of inputs \( u(t) \) is also \( L^2(-\infty, 0] \). This is useful because the Fourier transform is a Hilbert space isomorphism from \( L^2(-\infty, \infty) \) to \( L_2 \) [62]. In particular, \( L^2(-\infty, 0] \) is mapped onto \( H^+ \), the set of vectors whose components are strictly proper and have no poles in \( Re\{s\} < 0 \). The partial fraction expansion of a component of such

\(^4\)In general, hybrid impedance/admittance matrix.
a vector would result in:

\[ u(s) = \sum_i \frac{\text{residue}_i}{s - s_i}, \text{ where } s_i = \sigma_i + j\omega_i \]

Because \( u(s) \) is in \( H_2^1 \), \( \sigma_i \geq 0 \). The inverse transform of each term in the sum would be of the form:

\[ u(t) = u_i e^{s_i t} \]

where \( u_i \) is, in general, complex, and \( \sigma_i \geq 0 \). All inputs can now be formed by linear combinations of inputs of this form. Thus, we will investigate passivity for inputs and outputs of the following form:

\[
\begin{align*}
    u(t) &= u_0 e^{s_0 t}, \quad s_0 = \sigma_0 + j\omega_0, \quad \sigma_0 \geq 0 \\
    y(t) &= Z(s_0) u_0 e^{s_0 t}
\end{align*}
\]

Passivity requires the following:

\[ Re \left\{ \int_{-\infty}^{t} u_0^H Z(s_0) u_0 e^{2\sigma_0 \tau} \, d\tau \right\} \geq 0 \]  \hspace{1cm} (2.25)

for all \( u_0 \), all \( t \), and all \( \sigma_0 \geq 0 \). Carrying out the integral leads to:

\[ \frac{1}{2\sigma_0} Re\{u_0^H Z(s_0) u_0\} e^{2\sigma_0 t} \geq 0 \] \hspace{1cm} (2.26)

and to:

\[ Re\{u_0^H Z(s_0) u_0\} \geq 0 \] \hspace{1cm} (2.27)

for all admissible \( u_0 \) and all \( \sigma_0 \geq 0 \). Equation 2.27 may be rewritten as follows:

\[
\frac{1}{2} \left\{ [u_0^H Z(s_0) u_0] + [u_0^H Z(s_0) u_0]^H \right\} \geq 0
\]

\[
   u_0^H [Z(s_0) + Z^H(s_0)] u_0 \geq 0
\] \hspace{1cm} (2.28)

Equation 2.28 will be satisfied so long as the matrix \( Z + Z^H \) is non-negative definite Hermitian for all \( \sigma_0 \geq 0 \). This is the definition of a \textit{positive matrix}. Because we will be concerned with physical systems only, it will always be the case that \( Z \) is a \textit{real matrix}, i.e., that \( Z(s) \) is real for real \( s \). Thus, the input/output passivity criterion is:
Passivity 5a (Linear n-Port). A linear time-invariant n-port is passive iff \( Z + Z^H \) is a positive real matrix.

In order to explore this criterion further, consider now the case of 1-port systems, i.e., systems for which \( u \) and \( y \) are scalars. In this case, the passivity condition reduces to the following:

**Passivity 6a (Linear 1-Port).** A linear time-invariant 1-port is passive iff:

\[
\text{Re}\{Z(s)\} \geq 0 \text{ for } \sigma \geq 0
\]  
(2.29)

A function \( Z(s) \) which is a real, rational function of \( s \) and which satisfies this condition, is called a positive real function. This criterion, although rather simple, can be computationally complex because it requires evaluation of \( \text{Re}\{Z(s)\} \) at each point in the right half \( s \)-plane. It can be readily shown [13], however, that the following three conditions are equivalent to equation 2.29.

**Passivity 6b (Linear 1-Port).** A linear time-invariant 1-port is passive iff:

1. \( Z(s) \) has no poles in the right half plane.
2. Any imaginary poles of \( Z(s) \) are simple, and have positive real residues.
3. \( \text{Re}\{Z(j\omega)\} \geq 0. \)

These conditions are essentially the result of an application of the principle of maximum modulus [85] to a contour which encloses the region over which \( \sigma \geq 0 \), i.e., the right half \( s \)-plane. In order to apply this principle, however, the contour cannot enclose any poles; this is the reason for condition 1. Right half plane poles (and zeros) are, in fact, prohibited by equation 2.29, because \( \text{Re}\{Z(s)\} \) must take on negative values in the neighborhood of any singularity. Imaginary poles are acceptable so long as the negative real part occurs to the left of the pole only [13]; this is the reason for condition
2. The contour, however, must be indented to avoid any imaginary poles, thus it is just the familiar Nyquist contour. As a result, it is possible to combine conditions 2 and 3, and to arrive at the following statement of passivity:

**Passivity 6c (Linear 1-Port).** A linear time-invariant 1-port is passive iff:

1. \( Z(s) \) has no right half plane poles.

2. \( Z(s) \) has a Nyquist plot which lies wholly within the closed right half plane.

An interesting consequence of passivity which is illustrated by this statement is that the phase of \( Z(s) \) must lie between \(+90^\circ\) and \(-90^\circ\). The phase of \( Z(s) \) is, of course, the difference between the phase of the output waveform \( y(s) \) and the phase of the input waveform \( u(s) \). Consider the case in which both are pure sinusoids. If the phase is \( 0^\circ \), the average value of the product \( u(t)y(t) \) is positive, so that the system consumes energy. If the phase is \( \pm 90^\circ \), the average is zero, and the system is called lossless\(^5\). If the phase is \( < -90^\circ \) or \( > +90^\circ \), the average value is negative, and the system must produce energy.

One may wonder if conditions similar to those above can be found in the \( n \)-port case. In fact, they can. Consider again the \( n \)-port passivity criterion: \( Z(s) + Z^H(s) \) is non-negative definite Hermitian for all \( \sigma \geq 0 \). As in the 1-port case, a set of equivalent conditions can be found [4]:

**Passivity 5b (Linear \( n \)-Ports).** A linear time-invariant \( n \)-port is passive iff:

1. \( Z(s) + Z^H(s) \) has no poles in the right half plane.

2. Any imaginary poles are simple and have positive real residue matrices.

3. \( Z(j\omega) + Z^H(j\omega) \) is non-negative definite Hermitian.

\(^5\)for further discussion of losslessness, see [4].
Given these conditions, \( Z + Z^H \) need be tested only for right half plane poles and along the imaginary axis, following an indented contour around any imaginary poles. Because a non-negative definite Hermitian matrix must have zero or positive eigenvalues, one test would be to plot these eigenvalues as functions of \( \omega \).

A computationally less expensive test, however, would be to plot the determinants of the \( n \) upper left submatrices of \( Z + Z^H \) as functions of \( \omega \). Another consequence of the non-negative definite Hermitian property is that these determinants must all be zero or positive [77]. Finally, imaginary poles may be treated by applying the same test to their residue matrices, rather than by evaluating these determinants along an indentation.

To complete this tour through input/output passivity criteria, we will return to the single interaction port case and point out several corollaries of the passivity criterion which provide some useful insights into the nature of positive real functions. These are:

**Corollary 1.** The relative degree of \( Z(s) \) may be only \(-1, 0, \) or 1 (as implied by the previously mentioned phase angle condition).

**Corollary 2.** \( 1/Z(s) \) is also positive real; thus, like the poles, the zeros of \( Z(s) \) must lie in the closed left half plane.

**Corollary 3.** The zeros of \( Z(s) \) on the \( j\omega \) axis (including those at 0 and \( \infty \)) must be simple with positive differential quotients; i.e., for a zero at \( j\omega_i \):

\[
\lim_{s\to j\omega_i} \left( \frac{Z(s)}{s-j\omega_i} \right) \geq 0
\]  

(2.30)

Corollary 1 may be interpreted to mean that, at high frequency, any positive real driving point impedance must appear to be the driving point impedance of a mass, a spring, or a viscous damper (to use mechanical elements). Corollary 2 indicates that a driving point impedance may not exhibit nonminimum phase behavior. Corollary 3 is a useful tool to have in a visual inspection of a driving point impedance function;
PID controllers, for instance, often lead to closed-loop driving point impedances which violate this condition.

2.4.2 Network Synthesis

The purpose of this section is to introduce the idea of network synthesis, which will be a useful preliminary to the state-space criteria of the next section, as well as an essential notion in Chapter 6. The various passivity criteria which are being presented in this chapter are found almost exclusively in the electrical engineering literature; in particular, in the network analysis and synthesis literature. Various analytical tools, such as tests for positive realness, are typically intended to serve as preliminaries to the real problem of interest, which is: given an impedance which is a prescribed function of frequency, find an electrical network which has that behavior. This is the network synthesis problem.

One of the seminal results in this field concerns the synthesis of passive networks [13]:

If a driving point impedance function is positive real, then at least one network composed of passive, linear elements \((I, C, R, TF, GY)\) exists which exhibits that impedance.

Of course, this result can be generalized to positive real driving point impedance matrices. This theorem will not be proven here; however, it is motivated in Chapter 6 with a discussion of network synthesis procedures.

This result has certain consequences of immediate benefit. One of these is that any system with a positive real driving point impedance, even if it is feedback-controlled, has an equivalent representation, as seen at the ports, which is composed of purely passive, linear elements. This concept lies at the heart of the developments in Chapter 6.
The second consequence stems from the first. Because a passive, linear system always corresponds to some realization composed of passive, linear elements, the power flowing into the ports may be accounted for exactly as:

\[ \text{Power}_{\text{in}} = \frac{d}{dt} E_{\text{stored}} + \frac{d}{dt} E_{\text{dissipated}} \]

where energy is stored in \( I \) elements (e.g. masses, inductors) and \( C \) elements (e.g. springs, capacitors), and dissipated through \( R \) elements (e.g. viscous dampers, ohmic resistors). The utility of this statement will soon be evident.

### 2.4.3 State-Space Passivity Criteria

The state-space passivity criteria will be developed in the time domain in terms of the state and output equations 2.13. To develop the necessary conditions for passivity, we will assume that, as explained in the previous section, any passive linear system can be represented by some collection of linear elements. A power balance can, therefore, be written:

\[ u'(t)y(t) = \frac{d}{dt} (E_{\text{stored}}) + \frac{d}{dt} (E_{\text{dissipated}}) \quad (2.31) \]

The reader might be concerned that this is not a legitimate starting point as it apparently requires some \textit{a priori} understanding of the relationship between passivity and system structure. This is, in fact, true, but is a rather fine point that should not detract from an intuitively appealing derivation. A derivation similar to the one presented here, as well as several more rigorous derivations of state-space passivity criteria, can be found in [4].

For a linear system of minimal dimension, \( E_{\text{stored}} \) and \( \frac{d}{dt} E_{\text{dissipated}} \) may be represented as follows:

\[ E_{\text{stored}} = \frac{1}{2} x'Px \quad (2.32) \]
\[ \frac{d}{dt} E_{\text{dissipated}} = (\Phi x + \Gamma u)'R(\Phi x + \Gamma u) \quad (2.33) \]
where $P$ is a positive definite (PD) matrix, and $R$ is a positive definite diagonal matrix.

The equation for $E_{\text{stored}}$ should seem reasonable because the energy in a linear storage element (a spring or a mass) may be represented as a quadratic function of state; however, it can be motivated further by an argument based on bond graph modeling. The crux of the argument is that a bond graph model of a minimal realization of order $k$ will contain exactly $k$ independent storages [6]. We may reflect each of these storages through an appropriately chosen transformer so that each of the reflected storages has unity magnitude ($I = 1$ or $C = 1$), and select as state variables the flows on the inertias ($f_i^*$) and the efforts on the capacitances ($e_i^*$):

$$x^* = \begin{bmatrix} f^* \\ e^* \end{bmatrix}$$

Because the storages in the system are all independent, the total energy is given by the sum of the energies in the individual storages:

$$E = \frac{1}{2} x^* x^*$$

Now we appeal to the fact that any two state representations of a linear system can be related by a similarity transform. Thus, a nonsingular matrix $T$ exists which satisfies:

$$T x = x^*$$

We can use this transform to express the stored energy in terms of the original states:

$$E = \frac{1}{2} x' T' T x$$

If we define $P = T' T$, $P$ is clearly a positive definite matrix, and equation 2.32 is restored.

The equation for $\frac{d}{dt} E_{\text{dissipated}}$ is simply the sum of the power lost to each of the dissipative elements in the system. The flow into each dissipator is a linear function of the states and the inputs:

$$f_i = \phi x + \gamma u$$
and,
\[
\frac{d}{dt} E_{\text{dissipated}}^i = r_i f_i^2
\]
where \( r_i \) is the viscous coefficient (resistance) of the \( i^{th} \) element. If \( f_r \) is a column vector of flows into the system’s dissipators, then:
\[
f_r = \Phi x + \Gamma u
\]
and,
\[
\frac{d}{dt} E_{\text{dissipated}} = f_r' R f_r
\]
where \( R \) is a diagonal matrix of the \( r_i \). Thus, equation 2.33 is restored.

The power balance may now be written as follows:
\[
u'y = \frac{1}{2} \frac{d}{dt} x' P x + (\Phi x + \Gamma u)' R (\Phi x + \Gamma u)
\]
(2.34)
Taking the derivatives and using the state and output equations 2.13:
\[
u'C x + u'D u = \frac{1}{2} x' A' P x + \frac{1}{2} u' L' P x + \frac{1}{2} x' P A x + \frac{1}{2} x' P L u
\]
\[
+ x' \Phi' R \Phi x + u' \Gamma' R \Phi x + x' \Phi' R \Gamma u + u' \Gamma' R \Gamma u
\]
\[
u'C x + u'D u = x \left( \frac{1}{2} P A + \frac{1}{2} A' P + \Phi' R \Phi \right) x + u' \left( L' P + 2 \Gamma' R \Phi \right) x
\]
\[
+ u' \left( \Gamma' R \Gamma \right) u
\]
(2.35)
Because this equation holds for all \( x \) and \( u \), the following equations must be satisfied:

\[
0 = \frac{1}{2} P A + \frac{1}{2} A' P + \Phi' R \Phi
\]
(2.36)
\[
C = L' P + 2 \Gamma' R \Phi
\]
\[
D = \Gamma' R \Gamma
\]

Now defining \( M = \Phi' R^{1/2} \) and \( W = R^{1/2} \Gamma \), these equations reduce to the following:
\[
PA + A' P = -2 M M'
\]
(2.37)
\[
PL = C' - 2 M W
\]
\[
D = W' W
\]
58
Very rarely in mechanical systems does one encounter a situation in which $D \neq 0$. Therefore, the version of these passivity criteria which is most frequently encountered is:

\[
PA + A'P = -Q \tag{2.38}
\]
\[
PL = C'
\]

where $Q$ is positive semidefinite (PSD).

The sufficiency of equations 2.37 in establishing passivity can be demonstrated as follows. For any matrices $M$ and $W$:

\[
(M'x + Wu)'(M'x + Wu) \geq 0 \tag{2.39}
\]
\[
x'MM'x + 2u'W'Mx + u'W'Wu \geq 0 \tag{2.40}
\]

With the aid of equations 2.37, equation 2.40 may be rewritten as follows:

\[
-\frac{1}{2}x'(PA + A'P)x + u'(C - L'P)x + u'Du \geq 0 \tag{2.41}
\]
\[
u'Cx + u'Du \geq \frac{1}{2}x'PAx + \frac{1}{2}x'A'Px + \frac{1}{2}u'L'Px + \frac{1}{2}x'PLu \tag{2.42}
\]
\[
u'y \geq \frac{1}{2} \frac{d}{dt}(x'Px) \tag{2.43}
\]

Because $\frac{1}{2}x'Px$ can serve as an internal energy function (it need not necessarily represent the energy in the storage elements), this establishes passivity by equation 2.15. In summary:

**Passivity 7a (Linear n-Ports).** A linear, time-invariant $n$-port described by equations 2.13 is passive iff the following equations are satisfied for some PD $P$ and some $M$ and $W$:

\[
PA + A'P = -2MM' \]
\[
PL = C' - 2MW \]
\[
D = W'W
\]
Passivity 7b (Linear n-Ports). A linear, time-invariant n-port described by equations 2.13 with $D = 0$ is passive iff the following equations are satisfied for some PD $P$ and for some PSD $Q$:

$$PA + A'P = -Q$$
$$PL = C'$$

The fact that, for a 1-port, the equations 2.37 are satisfied if and only if a positive real function $Z(s) = D + C(sI - A)^{-1}L$ exists, is known as the Kalman-Yacubovitch Lemma, and has received widespread application in the study of absolute stability (see, for instance, Narendra and Taylor [61]). The generalization to n-ports is known as the Positive Real Lemma [4].

2.5 An Alternative Criterion for Positive Real Functions

The purpose of this section is to present an alternative set of necessary and sufficient conditions for positive realness, and, equivalently, an alternative criterion for the passivity of a 1-port. Although this criterion was independently developed for applications relevant to this thesis, it may be derived in a very simple fashion from one proposed by Talbot [79,80]. A physically motivated explanation of this criterion, as well as an application to the stability of coupled dynamic systems, is presented in Section 3.3.

A surprisingly large number of alternative tests for positive real functions have been reported in the literature. The more familiar of these were presented in the last section; one good source of others is Van Valkenburg [85]. Among the others are the Sturm test, which involves the construction of a Routh-like array, and a test due to Talbot which employs a conformal map to transform what is essentially a criterion on phase
to a criterion on magnitude (Anderson performs a similar transformation with positive real matrices [4]).

More relevant to the current development, however, are tests due to Brockett [12] and Talbot [79,80]. Brockett's test is the following:

If \( Z(s) = N(s)/D(s) \), where \( N(s) \) and \( D(s) \) are polynomials with real coefficients and without common factors, \( Z(s) \) is positive real iff \( N^2(s) + kD^2(s) \) is Hurwitz for all \( 0 < k < \infty \).

where Hurwitz means that the polynomial has no right half plane (RHP) zeros and any imaginary zeros are simple. The assumption that the polynomials have no common factors is equivalent to previous assumptions of minimality. Part of the value of this condition is that it is equivalent to the requirement that all branches of the root locus of \( Z^2(s) \) lie in the left half plane (LHP). Thus, it provides a simple, graphical test for positive realness.

Approximately one and a half years after Brockett published the theorem above, Talbot published the following similar theorem:

If \( Z(s) = N(s)/D(s) \), where \( N(s) \) and \( D(s) \) are polynomials with real coefficients and without common factors, \( Z(s) \) is positive real iff \( k_1N(s) + jk_2D(s) \) has no RHP zeros for all real constants \( k_1 \) and \( k_2 \), not both zero.

This theorem also affords a graphical test, moreover, one that relates directly to the driving point impedance of interest, \( Z(s) \). This test is simply that the locus of points where \( \arg(Z(s)) = \pm \pi/2 \) must lie in the LHP. This test has the disadvantage, however, that it does not employ so ubiquitous an analytical tool as root locus.

A minor variant of Talbot's condition, however, both applies directly to transfer functions of interest, (see Section 3.3), and takes advantage of root locus techniques. The following theorem leads to this test:

61
Passivity 8 (Linear 1-Ports). If \( Z(s) = N(s)/D(s) \), where \( N(s) \) and \( D(s) \) are polynomials with real coefficients and without common factors, \( Z(s) \) is positive real iff the polynomials \( r_1(s) = sD(s) + kN(s) \) and \( r_2(s) = D(s) + ksN(s) \) have no RHP zeros for all \( 0 \leq k < \infty \).

The equivalent root locus test is that all branches of the loci of \( sZ(s) \) and \( Z(s)/s \) must lie in the closed LHP. This theorem is evidently little more than a variant on the one proposed by Talbot; it is given special attention here because of the previously mentioned physical motivation. The proof of this theorem is as follows:

The method of proof is to demonstrate that the criterion above is equivalent to the three conditions which constitute Passivity 6b (page 52). This may be done as follows:

- For \( k = 0 \), \( r_2(s) = D(s) \), thus \( D(s) \) has no RHP zeros. This establishes condition 1 of Passivity 6b. Furthermore, as \( k \to \infty \), \( r_1(s) \to kN(s) \), thus \( N(s) \) has no RHP zeros.

- For all \( \sigma > 0 \), almost all \( \sigma = 0 \), all real \( \omega \), and all \( k \geq 0 \):

\[
\frac{N(s)}{D(s)} \neq \frac{-s}{k}, \quad s = \sigma + j\omega
\]

\[
\frac{N(s)}{D(s)} \neq \frac{1}{ks}
\]

These two equations have the following consequences:

\[
Re\{Z(\sigma \pm j\omega)\} > 0 \quad \text{for all } \sigma > 0 \quad (2.44)
\]

\[
Re\{Z(\sigma \pm j\omega)\} \neq 0 \quad \text{for almost all } \sigma = 0 \quad (2.45)
\]

Consider now the locus of points, shown in Figure 2.5, defined by \( \varepsilon + j\omega \), \( 0 \leq \varepsilon \ll 1 \), \( -\infty < \omega < \infty \), except for a finite number of right half plane indentations of radius \( 0 \leq \delta \ll 1 \) around the imaginary roots of \( D(s) \).

Equations 2.44 and 2.45 require that \( Re\{Z(j\omega)\} \geq 0 \) in the limit as \( \varepsilon \to 0 \). This establishes condition 3 of Passivity 6b.
Figure 2.5: Locus of points for evaluation of $Re\{Z(\sigma + j\omega)\}$.

- Along the indentations, $Re\{Z(s)\} > 0$. The behavior of $Z(s)$ near a singularity, however, can be determined in terms of the parameter $\theta$ as follows:

$$Re\{Z(s)\} = \frac{\text{Residue}}{\delta} \cos(-n\theta) \tag{2.46}$$

where $n$ is the multiplicity of the singularity. This expression will be greater than zero for all $\sigma > 0 \ (-90^\circ < \theta < 90^\circ)$ only if $n = 1$ and $\text{Residue} > 0$. This establishes condition 2 of Passivity 6b, and completes the proof.