Chapter 3

Coupled Stability

The purpose of this chapter is to address the issue of the stability of a feedback-controlled system which is *interacting with a dynamic environment*. As indicated in Section 1.4.1, the stability of a controlled system coupled to its environment is the most important consideration created by our interest in the design of interaction controllers. As such, it seems natural to achieve a solid understanding of coupled stability before attempting to resolve other design issues. The next section serves to review the previous art, as well as to develop the rationale for the approach taken here.

The term "coupled stability" will be used extensively in this chapter. Although it will eventually be defined more carefully, the reader should, at this point, take it to mean quite literally the stability of the system composed of the manipulator and environment, mechanically coupled together.

3.1 Approaches to Coupled Stability

It should come as no great surprise to the reader that the bulk of the robotics literature does not address interaction with the environment. The influence of control theory is evident in that, if an environment is considered at all, it is typically treated as a source of disturbances rather than a dynamic system in its own right. Recently, however, an
increasing number of researchers have taken note of the importance of environment dynamics in overall system behavior (where the system is taken to be the robot and its environment). For instance, even if the environment is assumed to be a rigid body, it may still be useful from the point of view of control (particularly computed torque methods) to know the inertial parameters of that rigid body. Thus, several approaches to load determination, assuming rigid body loads, have been developed [9,51].

Although inertial load identification presumes stability and is intended to address performance, coupling to dynamic environments has been known to endanger robot stability. The most compelling example of current interest is the instability of force-feedback controlled robots upon contact with rigid surfaces. “Contact instability” will be analyzed in detail in Chapter 7; for now, it is simply noted that the environment is a stiff surface, which many have modeled as a stiff spring.

Most of the efforts which treat the environment as a rigid body or a linear spring have command following (trajectory or force control) rather than interactive behavior as their primary goal, and, as such, make no effort to generate a comprehensive analysis of coupled stability. Oddly enough, even those who have been concerned with the design of impedance controllers, i.e., controllers for manipulators which are intended to interact with a fairly broad class of environments, have been slow to address the coupled stability behavior of closed loop designs. Until recently, greater emphasis was placed on performance aspects, such as the “closeness” of the closed loop behavior to that of a target model. Such an approach, however, is contrary to the set of design specifications proposed in Section 1.4.2, and may well result in undesirable behavior upon implementation. This sort of pitfall is illustrated by several of the examples in Chapters 4 and 8.

The impedance control literature reveals the following four levels of concern for coupled stability: the coupled stability properties of the:
• target dynamics.

• closed loop system, assuming that the robot is composed of rigid links with torque sources at the joints.

• closed loop system, assuming that additional dynamic effects (e.g., actuator bandwidth limitations or transmission dynamics) are important, but are included in the manipulator model.

• closed loop system, while accounting for modeling uncertainties, such as unmodeled dynamics or parametric errors.

The robust coupled stability property of a given set of target dynamics is an important justification for pursuing impedance control. For instance, Hogan has analyzed a manipulator which exhibits the following behavior [33]:

$$F = K(X - X_0) + B(V)$$

where $X$ and $V$ are the manipulator position and velocity vectors, $X_0$ is a vector of desired positions, $F$ is a force imposed by the environment, $K(\cdot)$ is the force-displacement relation, and $B(\cdot)$ is the force-velocity relation. He has shown that, so long as $K(\cdot)$ is the gradient of some potential function, and $B(\cdot)$ is chosen so that this manipulator is capable of stably positioning an arbitrarily small mass, then the manipulator can interact stably with any dissipative environment describable in Hamiltonian terms. Extensions to this analysis can be found in Fasse [22]. Kazerooni [44] makes a similar argument for the case in which both the manipulator and the environment can be described as 2-$n$ order linear systems (e.g., $n$-link robots, linearized) with positive definite inertia, damping, and stiffness matrices.

Apparently this sort of behavior is quite desirable, but can it be realized with feedback control and an actual robot? If the robot has rigid links and actuators which
act as pure torque sources, the answer is yes. Hogan has presented a controller which will, under these conditions, implement the target dynamics above (with an added inertia term), and even compensate for joint friction and gravity [35]. Takegaki and Arimoto also present a rigid-link, ideal-actuator robot controller which exhibits a strong coupled stability property [78].

In practice, however, these controllers do not necessarily succeed in implementing the robust coupled stability of the target dynamics (see, for instance, [92]). The reason is, of course, that real robots do not have such simple behavior as these methods assume. Actuator and transmission dynamics, joint and link flexibilities, and computational delays are but a few of the non-idealities. It turns out, however, that to account for these non-idealities, even if they are well characterized, can be quite difficult. For instance, in [44], Kazerooni performs a linearized analysis of an n-d.o.f. system with actuator bandwidth limitations. Each actuator is assumed to have a first-order rolloff. He is careful to point out that the closed loop system will be guaranteed to have the coupled stability property of the target dynamics only if the two have identical behavior at all frequencies, but that this cannot happen. The coupled stability properties of his designs are, in fact, rather poor—a detailed example is presented in Section 4.1.3.

In more recent work [45], Kazerooni has developed a method for guaranteeing the stability of closed loop designs of arbitrary complexity coupled to environments which are characterized by the magnitude of their input/output mapping (e.g., for linear systems, the singular values). This method is developed quite naturally for linear systems using the techniques of multivariable analysis; he has also shown that it can be extended via the Small Gain Theorem to apply to a broad range of nonlinear systems. This technique will be analyzed in some detail in Chapter 8; for the present purpose, it is noted that the important benefit of Kazerooni's method is that it provides an analysis for a very general set of systems and even (although he doesn't treat this) provides a framework suitable for treating modeling uncertainties; however, it has the
drawback that the magnitude (e.g., stiffness) of the environment must be bounded. This is inconvenient in, for instance, the case of a rigid surface.

Fasse [22] has developed a method for guaranteeing the stability of closed-loop systems coupled to passive environments (which may be nonlinear). The basis of his method is the selection of an appropriate Lyapunov function for the controlled system. An important result of this work is that the Lyapunov function, \( V \), need not represent an actual energy so long as the following relation holds:

\[
\dot{V} = -D + u'y
\]

where \( D \) is a positive semi-definite function of state, and \( u'y \) is the power delivered to the ports of the system. In the terminology of the last chapter, \( V \) is an internal energy function. Fasse shows several examples of linear systems for which such Lyapunov functions can be found by restricting the feedback gains in appropriate ways.

If such a Lyapunov function can be found, it constitutes a sufficient, but not necessary condition for the stability of the closed loop system coupled to arbitrary passive environments. In the next section, a necessary and sufficient condition will be found; thus, Fasse's result is a subset of the one found here.

Fasse also addresses the robustness of coupled stability to various types of modeling errors. In these analyses, he treats the standard implementations proposed by Hogan [35,92], but subject to non-idealities such as sensor gain errors, actuator dynamics, base dynamics, friction, and gravity. His is the only systematic examination of coupled stability robustness available to date.

This thesis will not add to his results, but will focus on the criteria which must be satisfied by any physical system, controlled or not, if it is to retain stability when interacting with arbitrary passive environments. However, so that the analysis is manageable, only linear plants will be considered.

The restriction to passive environments merits some further explanation. There are,
in fact, several reasons for focusing on passive environments. The first of these is that many environments of interest are passive. Rigid bodies, rigid surfaces, springs, and any collection of these are passive. But many other systems, less easily modeled and categorized, are also passive. For instance, a crucible of molten iron is passive. A second reason is that passive systems are mathematically well-defined and understood; this was the subject of the last chapter. Furthermore, it can be argued that the restriction of the environment to be a passive system is a much more natural choice than the restriction of the environment to be a system of certain input/output magnitude bounds. Certainly it is convenient for a manipulator to be able to interact with both inertial loads and rigid surfaces. Finally, the end justifies the means. The coupled stability analysis of the next section yields significant insight into the general topic of coupled dynamic systems, so that a number of extensions to the basic analysis are possible. For instance, in Section 3.5 it is shown that the stability criterion also applies to a broad class of active systems.

Before moving on to the coupled stability criterion, the term “coupled stability” will be more carefully defined. For this document, the definition given by Fasse [22] will be used:

**Coupled Stability.** A system is said to have the coupled stability property if:

1. The system is stable when isolated.
2. The system remains stable when coupled to any passive environment which is also stable when isolated.

The reader may be concerned about the common situation in which the manipulator and environment are in contact, but are not mechanically coupled. The relation to “contact stability” will be discussed in Section 3.6.
Figure 3.1: (a) Isolated systems, each with an available interaction port. (b) Coupled system. An alternative would have been to couple these systems with a 0 junction, in which case $f_a = f_b$ and $v_a = -v_b$; however, the stability result would be unaffected. See [68] for an explanation of 0 and 1 junctions.

3.2 1-Port Interaction

The purpose of this section is to derive a set of necessary and sufficient conditions for the stability of a linear, time-invariant system coupled at a single interaction port to an arbitrary linear, passive environment. The system in question (which may also be referred to as the plant) may be feedback controlled or not; the environment is assumed to be passive according to Passivity 6c of Section 2.4.1.

To commence, consider two linear, time-invariant systems, each with an open interaction port, as indicated in Figure 3.1(a). We are interested in the stability of the system that results when these are coupled at their interaction ports, i.e. the stability of the system shown in Figure 3.1(b).

Figure 3.1(b) can be rewritten in the block diagram form shown in Figure 3.2.
Implicit in this representation is the linear nature of $A$ and $B$, i.e., both $A(s)$ and $B(s)$ may be expressed as the ratio of two rational polynomials in $s$. Furthermore, because each of these transfer functions relates power variables ($f$ and $v$) measured at a given interaction port, they are driving point impedances.

The dynamics of the coupled system may be expressed as follows:

$$0 = \begin{bmatrix} -1 & B(s) \\ -A(s) & -1 \end{bmatrix} \begin{bmatrix} v(s) \\ f(s) \end{bmatrix}$$  \hspace{1cm} (3.1)

The roots of the characteristic equation, $1 + A(s)B(s) = 0$, must fall within the closed LHP if a stable solution is to exist.

A Nyquist procedure can be used to study the pole locations. The basis of the Nyquist procedure is the use of the Principle of the Argument and of a well chosen mapping between complex planes. The Principle of the Argument states:

The number of clockwise encirclements of the origin by a map through $G(s)$ of a clockwise contour in the $s$ plane equals the number of zeros of $G(s)$ within the contour minus the number of poles of $G(s)$ within the contour.

A counterclockwise encirclement counts as minus one clockwise encirclements.

In classical control theory, the transfer function of interest is generally $1 + G(s)$ rather than $G(s)$. This is because, assuming unity feedback and $G(s) = N(s)/D(s)$ as
an open loop transfer function, \(1 + G(s)\) is the quotient of the closed loop characteristic polynomial \((N(s) + D(s))\) and the open loop characteristic polynomial \((D(s))\). Rather than map through \(1 + G(s)\), however, it is convenient to map through \(G(s)\) and to investigate encirclements of the \(-1\) point.

The Nyquist contour, which is shown in Figure 3.3, includes the entire right half plane, but excludes any poles on the imaginary axis. An example of a mapping into the \(G(s)\) plane is also shown in this figure. This mapping is necessarily symmetric about the real axis as \(G(s)\) is the ratio of two polynomials in \(s\), \(N(s)/D(s)\), the real part of which is an even function of \(\omega\) on the \(j\omega\) axis, and the imaginary part of which is an odd function. The behavior very near a pole on the \(j\omega\) axis is determined by this pole (or poles, if they are not simple) and its residue; thus, an arc of 180° in the \(s\) plane, taken at an infinitesimal radius around an imaginary pole, and starting at \(\pm 90°\), must result in an arc in the \(G(s)\) plane that is some multiple of 180°, and is symmetric about the real axis. Similarly, the behavior along the arc of radius \(R\), as \(R \to \infty\), is determined by \(\alpha s^n\), where \(\alpha\) is a constant, and \(n\) is the relative order of \(G(s)\); thus, the mapping of this arc must also be symmetric about the real axis of the \(G(s)\) plane.

Once the mapping through \(G(s)\) has been generated, it is a simple matter to interpret the Principle of the Argument as follows:

The number of clockwise encirclements of the \(-1\) point by a mapping of the Nyquist contour through \(G(s)\) equals the number of unstable closed loop poles (i.e., RHP roots of \(N(s) + D(s)\)) minus the number of unstable open loop poles (i.e., RHP roots of \(D(s)\)).

Figure 3.2 indicates that \(A(s)B(s)\) plays the role of the open loop transfer function; thus, we should replace \(G(s)\) with \(A(s)B(s)\) in the statement above. Some further assumptions about \(A(s)\) and \(B(s)\) are now necessary. First \(A(s)\) is identified with the plant, and \(B(s)\) with the environment.
Figure 3.3: An example mapping of the Nyquist contour.

It will also be assumed that the input and output of $A(s)$ have been chosen appropriately so that a state space realization exits; i.e., so that $F_a$, $G_a$, $H_a$, and $J_a$ exist which satisfy:

\[ \dot{x} = F_a x + G_a v \]  
\[ f = H_a x + J_a v \]

and:

\[ A(s) = J_a + H_a(sI - F_a)^{-1}G_a \]

This is equivalent to the requirement that $A(\infty) < \infty$. It is further assumed that the eigenvalues of $F_a$ lie in the closed left half plane. These assumptions, taken together, guarantee that the poles of $A(s)$ lie in the closed left half plane, and are indicative of a stable system.

The environment, which was previously assumed to be passive, must satisfy the following criteria (from Passivity 6c, Section 2.4.1):
• $B(s)$ has no RHP poles.

• $B(s)$ has a Nyquist plot which lies wholly within the closed right half plane.

Otherwise stated (see Section 2.4.1), there is no restriction on the magnitude of $B(s)$, but the phase of $B(s)$ must lie between $-90^\circ$ and $+90^\circ$.

Recall that we are interested in guaranteeing the stability of the plant when coupled to any linear, passive environment. If $B(s)$ is viewed as a compensator which affects the magnitude and phase of $A(s)$, then the class of all such compensators will include those capable of altering the phase by $\pm 90^\circ$, and changing the magnitude by a factor of 0 to $\infty$. Therefore, it is clear that, in order to interact with all passive environments, $A(s)$ must have a phase margin of $\pm 90^\circ$, an upward gain margin of $\infty$, and a downward gain margin of 0.

The effect of these restrictions is to limit the mappings of the Nyquist contour through $A(s)$ to lie in the closed right half plane. For any path that enters the left half plane in a clockwise fashion, no more than $90^\circ$ additional lead or lag will be necessary to cause the path to cross the negative real axis, and, once crossing the axis, some gain between 0 and $\infty$ will cause the path to encircle the $-1$ point. An example of this is illustrated in Figure 3.4(a); the Nyquist plots shown in Figures 3.4(b) and (c) are clearly inadmissable for the same reason. The Nyquist plots shown in Figures 3.4(d) and (e), however, enter the left half plane in a counterclockwise fashion, and will not be reshaped by any acceptable $B(s)$ to yield a clockwise encirclement of the origin. It may be shown, however (see Appendix A), that any Nyquist plot which contains a counterclockwise loop is indicative of a RHP pole, violating one of our previous assumptions.

Thus, the mapping through $A(s)$, like that through $B(s)$, is restricted to lie wholly within the closed right half plane. This, taken together with the stability of $A(s)$ in isolation, is a necessary and sufficient condition to ensure coupled stability. It is also,
Figure 3.4: (a) Nyquist plots of a plant and a passive environment which, when coupled, lead to instability. (b),(c) Nyquist plots which fail to meet the coupled stability criterion. (d),(e) Nyquist plots which apparently cannot lead to coupled instability, but are indicative of RHP plant poles.
according to Passivity 6c, necessary and sufficient for $A(s)$ to represent the driving point impedance of a passive system. Note that this does not prohibit the plant, $A$, from being feedback-controlled; examples of feedback-controlled systems which obey this criterion are presented in the next chapter. The result of this analysis may be summarized as follows:

**Coupled Stability 1 (1-Port).** A necessary and sufficient condition to ensure the stability of a LTI, stable plant coupled at a single port to a LTI, stable, passive environment, is that the driving point impedance of the plant be positive real; or, equivalently, that the plant be passive.

### 3.3 $n$-Port Interaction

The purpose of this section is to derive a necessary and sufficient condition for the stability of a linear, time-invariant system coupled at $n$ interaction ports to a linear, passive environment. Both the plant and environment will be assumed to be strictly proper impedances or admittances. This may seem restrictive, but one can always take a physical system, active or not, and, if it is to be described as an impedance, add an arbitrarily stiff spring to the interaction port, thereby making it strictly proper. The same can be done by adding an arbitrarily small mass to the interaction port of an admittance. It can also be argued that Nature takes care of this problem for us—that any physical system must be causal, and therefore, if linear, strictly proper. Thus, the restriction to strictly proper systems does not weaken the proof.

Consider two linear, time-invariant $n$-ports, A and B:

\[
\begin{align*}
\dot{x}_a &= A_a x_a + L_a e_a \\
\mathbf{f}_a &= C_a x_a
\end{align*}
\]
\[ \dot{x}_b = A_b x_b + L_b f_b \]
\[ e_b = C_b x_b \]

where \( e \) and \( f \) represent power duals along the interaction ports of the two systems, and \( e_a, e_b, f_a, \) and \( f_b \) are all \( n \times 1 \) vectors. \( A_a \) is a \( k \times k \) matrix, and \( A_b \) is an \( m \times m \) matrix; \( k \) and \( m \) are not necessarily equal. Now consider the following coupling of \( A \) and \( B \):

\[ \begin{align*}
e_a &= -e_b \\
f_a &= f_b
\end{align*} \]

This may be thought of as coupling \( A \) and \( B \) with a 1 junction.

The dynamics of the coupled system are represented by the following state equations:

\[ \begin{bmatrix} \dot{x}_a \\ \dot{x}_b \end{bmatrix} = \begin{bmatrix} A_a & L_a C_b \\ -L_b C_a & A_b \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} \]  \hspace{1cm} (3.4)

which will, for notational convenience, sometimes be referred to as:

\[ \dot{x}_c = A_c x_c \]  \hspace{1cm} (3.5)

Necessary and sufficient conditions for the stability of the coupled system are the existence of PD \( P \) and PSD \( Q \) that satisfy:

\[ P_c A_c + A_c' P_c = -Q_c \]  \hspace{1cm} (3.6)

which is simply the Lyapunov stability equation. For the remainder of the proof, \( P_c \) will be expressed in the following partitioned form:

\[ P_c = \begin{bmatrix} P_1 & R \\ R' & P_2 \end{bmatrix} \]  \hspace{1cm} (3.7)

where \( P_1 \) is a \( k \times k \) matrix, \( P_2 \) is an \( m \times m \) matrix, and \( R \) is a \( k \times m \) matrix.
Some further assumptions about A and B are now necessary. The first is that A is stable in isolation. A PD matrix $P_a$ and PSD matrix $Q_a$ exist which satisfy the Lyapunov stability equation:

$$P_a A_a + A_a' P_a = -Q_a$$  

(3.8)

The second assumption is that system B represents a passive environment, in which case Passivity 7b applies:

$$P_b A_b + A_b' P_b = -Q_b$$  

(3.9)

$$P_b L_b = C_b'$$  

(3.10)

With these preliminaries completed, the proof will proceed in two parts, sufficiency and necessity.

**Sufficiency Proof**

Suppose that $P_1 = P_a$, $P_2 = P_b$, and $R = 0$, then:

$$P_c A_c + A_c' P_c = \begin{bmatrix} P_a & 0 \\ 0 & P_b \end{bmatrix} \begin{bmatrix} A_a & L_a C_b \\ -L_a C_a & A_b \end{bmatrix} + \begin{bmatrix} A_a' \\ C_a' L_a' \end{bmatrix} P_a \begin{bmatrix} P_a & 0 \\ 0 & P_b \end{bmatrix}$$  

(3.11)

After multiplying, adding, and using equations 3.8 and 3.9:

$$P_c A_c + A_c' P_c = \begin{bmatrix} -Q_a & P_a L_a C_b - C_a' L_a' P_b \\ C_b' L_a' P_a - P_b L_b C_a & -Q_b \end{bmatrix}$$  

(3.12)

Equation 3.10 may be used to simplify this matrix:

$$P_c A_c + A_c' P_c = \begin{bmatrix} -Q_a & (P_a L_a - C_a') C_b \\ C_b' (L_a' P_a - C_a) & -Q_b \end{bmatrix}$$  

(3.13)

The condition $P_a L_a = C_a'$, which, by Passivity 7b, is sufficient to guarantee that system A is positive real, is clearly sufficient to establish the stability of the coupled system:

$$P_c A_c + A_c' P_c = -Q_c = \begin{bmatrix} -Q_a & 0 \\ 0 & -Q_b \end{bmatrix}$$  

(3.14)
Necessity Proof

Having demonstrated sufficiency, we now intend to show that \( P_c L_a = C_a' \) is a necessary condition for the existence of positive definite \( P_c \) and positive semidefinite \( Q_c \). To do this, we begin by expressing \( P_c A_c + A_c' P_c \) in terms of the partitioned \( P_c \) (equation 3.7):

\[
P_c A_c + A_c' P_c = \begin{bmatrix}
RL_b C_a + C_a' L'_b R' - (P_1 A_a + A'_1 P_1) & C'_a L'_b P_2 - P_1 L_a C_b - (A'_a R + RA_b) \\
-C'_a L'_b P_1 + P_2 L_b C_a - (R' A_a + A'_a R') & -C'_a L'_b R - R' L_a C_b - (P_2 A_b + A'_b P_2)
\end{bmatrix}
\]  
\[(3.15)\]

Because all of the submatrices along the main diagonal of a PSD matrix must be PSD, the diagonal terms in equation 3.15 must be PSD. These terms are sums, and, in general, the individual components of the sum need not be PSD. This brings us to a critical point in the proof. We are interested in guaranteeing the stability of system \( A \) when it may be coupled to any passive environment. Because a passive system must satisfy Passivity 7b, all we need do to construct the state equations of a passive system is select \( A_b \) to satisfy equation 3.9, and then select any \( L_b \) and find the associated \( C_b \) with equation 3.10 (or select \( C_b \) and find \( L_b \)). The point is that either \( L_b \) or \( C_b \) may be considered to be completely arbitrary.

This brings us back to the submatrices which must be PSD. Although, in general, the components in the sum need not be PSD, if \( L_b \) is taken to be completely arbitrary, then the term \( RL_b C_a' + C_a' L'_b R' \) found in the upper diagonal term of equation 3.15 must be PSD independent of \( L_b \) (otherwise, a simple scaling of any given \( L_b \) would be sufficient to force the entire term to be non-PSD). This will occur only if \( R \) has a particular form:

\[
R = C_a' L'_b J_1
\]  
\[(3.16)\]
where $J_1$ is a PSD matrix. Then:

$$C'_a L'_b R' + RL_b C_a = 2C'_a L'_b J_1 L_b C_a$$  \hspace{1cm} (3.17)

Thus, although $L_b$ is completely arbitrary, the left side of equation 3.17 is always PSD. Similar arguments may be made for the lower diagonal term in equation 3.15. The term $-C'_b L'_a R - R'L_a C_b$, which may be rewritten using equation 3.10 as $-P_b L_b L'_a R - R'L_a L'_b P_b$, must be PSD. This will occur only if $R$ has the following form:

$$R = -J_2 L_a L'_b P_b$$  \hspace{1cm} (3.18)

where $J_2$ is PSD. Then:

$$-P_b L_b L'_a R - R'L_a L'_b P_b = 2P_b L_b L'_a J_2 L_a L'_b P_b$$  \hspace{1cm} (3.19)

Equations 3.16 and 3.18 may be equated to solve for $J_1$ and $J_2$:

$$C'_a L'_b J_1 = -J_2 L_a L'_b P_b$$  \hspace{1cm} (3.20)

A trivial solution to this equation is $J_1 = 0$ and $J_2 = 0$, however, it is necessary to determine if there are any others. Pre-multiplying by $L'_a$, and post-multiplying by $L'_b$:

$$(L'_a C'_a)(L'_b J_1 L_b) = -(L'_a J_2 L_a)(L'_b P_b L_b)$$  \hspace{1cm} (3.21)

Because $L_b$ is arbitrary, it is clear that a nontrivial solution to equation 3.21 can be found in general only if $L_b$ can be removed from the equation. This will occur only if we select $J_1 = \Lambda P_b$, where $\Lambda$ is some PSD diagonal matrix. This done, equation 3.21 reduces to:

$$L'_a C'_a = -\Lambda^{-1}(L'_a J_2 L_a)$$  \hspace{1cm} (3.22)

Equation 3.22 will be satisfied only if the matrix $L'_a C'_a$ has negative or zero eigenvalues. However, we already know from the sufficiency proof that one solution is:

$$L'_a C'_a = L'_a P_b L_a$$

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This solution clearly requires that $L'_a C'_a$ have positive or zero eigenvalues. The only reconciliation, other than the trivial solution of $L_a = 0$ and $C_a = 0$, is that $J_1 = 0$ and $J_2 = 0$. $R$, therefore, must also be a zero matrix.

Now equation 3.15 may be written in a simplified form:

$$P_c A_c + A'_c P_c = \begin{bmatrix} -(P_1 A_a + A'_a P_1) & C'_a L'_b P_2 - P_1 L_a C_b \\ P_2 L_b C_a - C'_b L'_a P_1 & -(P_2 A_b + A'_b P_2) \end{bmatrix}$$  \hspace{1cm} (3.23)

Because the right hand side of this equation must be the negative of a PSD matrix whose submatrices along the main diagonal are PSD, the following equations must hold:

$$P_1 A_a + A'_a P_1 = -Q_1 = -V'_1 V_1$$  \hspace{1cm} (3.24)

$$P_2 A_b + A'_b P_2 = -Q_2 = -V'_2 V_2$$  \hspace{1cm} (3.25)

where $V_1$ and $V_2$ are unrestricted except in dimension, and $Q_1$ and $Q_2$ are PSD. Furthermore, one of the following two equations must hold:

$$\begin{bmatrix} Q_1 & P_1 L_a C_b - C'_a L'_b P_2 \\ C'_b L'_a P_1 - P_2 L_b C_a & Q_2 \end{bmatrix} = \begin{bmatrix} V'_1 \\ V'_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$  \hspace{1cm} (3.26)

or,

$$\begin{bmatrix} Q_1 & P_1 L_a C_b - C'_a L'_b P_2 \\ C'_b L'_a P_1 - P_2 L_b C_a & Q_2 \end{bmatrix} = \begin{bmatrix} V'_1 & 0 \\ 0 & V'_2 \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}$$  \hspace{1cm} (3.27)

Equation 3.26 requires the following:

$$P_1 L_a L'_b P_b - C'_a L'_b P_2 = V'_1 V_2$$

As with equation 3.21, $L_b$ must be removed from this equation if it is to have a general solution. This would require $P_2 = \Lambda P_b$, where $\Lambda$ is a PD diagonal matrix, and $V_2 = \Gamma L'_b P_2$, where $\Gamma$ is some matrix of appropriate dimension. Substituting this expression into equation 3.25 produces the following result, which clearly has no solution for arbitrary $L_b$:

$$P_2 A_b + A'_b P_2 = -P_2 L_b \Gamma' \Gamma L'_b P_2$$

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The second option requires the following:

\[ P_1 L_a L_b^t P_b - C'_a L_b^t P_2 = 0 \]

Again, \( P_2 = \Lambda P_b \) is required to remove \( L_b \) from this equation. This done, the following equation must hold:

\[ P_1 L_a - \Lambda C'_a = 0 \] (3.28)

If we define \( P_a = \Lambda^{-1} P_1 \), and \( Q_a = \Lambda^{-1} Q_1 \), then we may write the necessary conditions for coupled stability as follows:

\[
\begin{align*}
P_a A_a + A'_a P_a &= -Q_a \\
P_a L_a &= C'_a
\end{align*}
\] (3.29)

These equations constitute the necessary and sufficient conditions for system A to be passive, according to Passivity 7b. In summary, the following result has been reached:

**Coupled Stability 2 (n-Port).** A necessary and sufficient condition to ensure the stability of a LTI, stable plant coupled at \( n \) ports to a LTI, stable, passive environment, is that the plant be passive according to Passivity 7b.

### 3.4 The “Worst” Environment

The purpose of this section is to describe an alternative method of analyzing coupled stability in the case of 1-port interaction. It will be demonstrated that, if a manipulator is stable when coupled to a particular, but restricted, class of passive environments, the “worst” environments, then it is stable when coupled to all passive environments; and furthermore, that this restricted class of environments is easily parametrized so that coupled stability can be analyzed by the generation of two root loci.
If we return briefly to the intuitive treatment of passive environments as a class of compensators which can add up to $\pm 90^\circ$ phase lag and any magnitude scaling to the open loop transfer function (the plant's driving point impedance), then it would seem reasonable that the most destabilizing environments would be those adding the full $\pm 90^\circ$ phase lag and some appropriate magnitude scaling. It is lossless environments [4] which have $\pm 90^\circ$ phase lag. Lossless environments are composed purely of storage elements such as springs or masses, with no dissipation. In fact, the simplest lossless environments are simply masses and springs which have driving point impedances and admittances of the form $a/s$ or $as$.

The intuitive approach, then, would suggest that if a plant is stable when coupled to any mass or any spring, it is stable when coupled to any passive environment. In fact, this result can be formally established with the use of Passivity 8 from Section 2.5. Passivity 8 requires that all branches of the root loci of $sZ(s)$ and $Z(s)/s$ lie in the closed LHP if $Z(s)$ is to represent a passive system. Because Coupled Stability 1 (Section 3.2) requires that the driving point impedance, $A(s)$, of the plant be that of a passive system, it is evident that $sA(s)$ and $A(s)/s$ must also have loci which lie in the closed LHP.

Now consider the case in which $A(s)$ is a driving point admittance. The coupling of $A(s)$ to a spring of stiffness $k_s$ is shown in Figure 3.5. Evidently, the stability of the coupled system can be determined for all stiffnesses $k_s$ simply by generating a root locus for the open loop transfer function, $A(s)/s$. Similarly, if the plant is coupled to a mass, stability can be determined for all values of the mass simply by generating a root locus of $sA(s)$. These same results hold if $A(s)$ is a driving point impedance, except that stability when coupled to springs requires a root locus of $sA(s)$, and when coupled to masses, a locus of $A(s)/s$. In summary:

**Coupled Stability 3 (1-Port).** A LTI plant interacting with its environment at a
single port will have the coupled stability property iff it is stable when coupled to all springs and all masses. Equivalently, the root loci of $A(s)/s$ and $sA(s)$, where $A(s)$ is the plant's driving point impedance, must lie completely within the closed LHP to guarantee coupled stability.

This method will be used in some of the examples of the next chapter. One important caveat: the term "worst" environment is something of a misnomer; any environment which leads to instability need not be outdone, and in most cases, if a lossless environment leads to instability, many dissipative environments will as well. Furthermore, finding those springs and masses which lead to instability says nothing about any other environments which will lead to instability.

3.5 Extensions to the Coupled Stability Analysis

The coupled stability criteria which have been derived in this chapter would not be nearly so powerful as they actually are if the plant actually had to be passive. In fact, it does not. Figure 3.6 shows how an active term may be added without jeopardizing the stability of the coupled system. Clearly, so long as $u$ does not depend on the states of the plant or the environment, it does not affect the system stability.

Some further insight can be gained by realizing that the driving point impedance
Figure 3.6: The addition of an active term to the coupled system (the plant is described by equations 3.2, with the addition of $B_a u$).

treated so far is simply a Thevenin or Norton equivalent impedance seen at that port, and that the presence of a general Thevenin or Norton equivalent source does not change the stability result\(^1\). In fact, the source term can itself be the output of some dynamic system, so long as the behavior of this system does not depend on the states of the plant or environment. Finally, it is evident that the environment may be active as well, so long as it has the equivalent impedance of a passive system.

Another important extension to the basic result is the inclusion of nonlinear passive environments. If the environment is nonlinear, passive, and stable, then there must exist some Lyapunov function which is also an internal energy function. Because a linear, passive plant also has a Lyapunov function which will serve as an internal energy function, then the two may simply be summed, following the arguments of Fasse [22], to demonstrate the stability of the coupled system. This provides a sufficient condition for coupled stability; necessity follows from the linear analyses, as linear systems are a subset of nonlinear. Thus:

\(^1\)Anderson and Spong [5] have also used Thevenin and Norton equivalents to describe a robot and its environment.
Coupled Stability 4 (n-Port). A LTI n-port plant will be stable when coupled to an arbitrary passive environment iff it has the driving point impedance of a passive system.

3.6 Contact Stability

The relation of contact stability to coupled stability can now be understood in a straightforward manner. A system which can make or break contact with its environment is essentially operating between two modes—uncoupled and coupled. If it is stable in both modes, then it must be stable in general, as the contact cannot create energy. Thus, coupled stability is a sufficient condition for contact stability.

Now suppose that the plant is unstable when coupled to some environment. If it is not secured to the environment, then it must eventually lose contact. At this point the plant will return to stable behavior and begin to dissipate energy. However, assuming that its equilibrium configuration brings it once again into contact with the surface, limit cycles will begin. These limit cycles cannot die away because of the coupled instability, but they do not necessarily have to increase without bound, either. At some amplitude, the energy loss when uncoupled may balance the energy gain when coupled.

Thus, bounded-input bounded-output stability can sometimes be established for contact stability when the coupled stability condition indicates instability. This cannot necessarily be resolved by requiring asymptotic stability, either, as our statement of coupled stability allows lossless systems, which are not asymptotically stable.

One resolution, which has been used in this thesis, is simply to disallow sustained oscillations which include a change in mode. If such behavior is considered unstable, coupled instability is also a sufficient condition for contact instability.
3.7 Summary

The most important result of this chapter is a necessary and sufficient condition for coupled stability:

A LTI $n$-port plant will be stable when coupled to an arbitrary passive environment iff it has the driving point impedance of a passive system.

It has also been shown that the plant need not be passive. Arbitrary state-independent source terms can be introduced without affecting the stability result. This is the reason for the structure presented in Section 1.4.2, which separates state-dependent and state-independent source terms.

An alternative criterion for 1-port coupled stability, based on the generation of two root loci, one for coupling to springs and one for coupling to masses, has also been presented. Finally, it has been shown that coupled stability and instability are sufficient conditions for contact stability and instability, so long as sustained limit cycles which include loss of contact with the environment are considered unstable.