STRICKLY POSITIVE REAL ADMITTANCES FOR COUPLED STABILITY

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Abstract
This paper builds upon recent work that has addressed the stability of a feedback-controlled robot coupled to a passive, dynamic environment. A new definition of a "strictly positive real" function is presented, and is used to provide necessary and sufficient conditions for the exponential stability of a coupled system comprising a 1-port robot with a strictly positive real admittance, and a 1-port environment with a positive real (but otherwise arbitrary) impedance. The distinction between the new definition and conventional definitions of a strictly positive real function is founded in physical systems theory: the new definition relies upon distinct roles for efforts and flows, and upon the concept of an excess state. This definition will provide a useful design constraint for the development of robust robot controllers.
1. Introduction

Positive real (PR) and strictly positive real (SPR) functions have found wide application in network analysis [1], absolute stability [8], and model-referenced adaptive control [13]. Recently, the concepts of passivity and PR functions have been used to study the stability of energetic interaction between a robot and its environment (workpiece) [2] [7]. Instability upon contact with a stiff environment has proven to be one of the major impediments to the implementation of feedback controlled mechanical compliance [11].

In [7] it was proved that a necessary and sufficient condition for the stability of a linear, time-invariant 1-port coupled to an arbitrary passive environment (linear or nonlinear), is that the impedance or admittance of the 1-port be PR. Simply put, the 1-port (i.e., the robot1) must exhibit a passive admittance (or impedance) if its stability is to be guaranteed when it interacts with a passive, but otherwise arbitrary, environment. Moreover, if the 1-port does not exhibit a passive admittance, then there is some passive environment that, upon coupling, will destabilize it. It was also shown that at least one such environment is a linear, lossless mass or a linear, lossless spring. It should be recognized that, in robotics, environments that are adequately modeled as ideal masses or ideal springs are not uncommon. A significant weakness of these results, however, is that the passivity of the robot is sufficient only to guarantee that exponential rates of growth do not occur, but is not sufficient to guarantee either BIBO or asymptotic stability. In this paper, we will show that exponential stability of the coupled system (when the environment impedance is PR) can be guaranteed by requiring that the admittance of the robot be SPR. However, the appropriate definition of SPR differs in one significant respect from those that have previously been proposed.

In the next section, a problem formulation is presented including two formal problem statements. In Section 3, the shortcomings of conventional definitions of SPR are explained, and a solution to the second problem is presented. In Section 4, a distinction — crucial to the proposed definition of SPR — is made between "excess

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1 As robots are inherently nonlinear, multiport systems, the use of linear, 1-port analysis may be suspect. However, the primary cause of "contact instability" is noncolocation of actuators and sensors [12] [9], which is an effect that can be adequately represented with linear, 1-port models. Moreover, linear models provide a basis for quantitative analysis and for controller design.
Nomenclature

\((A,B,C,D)\) minimal realization of an impedance or admittance
\(A_{coupled}\) A-matrix of the coupled system
\(B\) viscous damping coefficient
\(D(s)\) denominator polynomial
\(f, F\) force
\(I_{n\times n}\) \(n\times n\) identity matrix
\(K\) stiffness
\(M\) mass
\(N(s)\) numerator polynomial
\(\mathbb{R}^{n\times m}\) set of \(n\times m\) real-valued matrices
\(s\) complex variable \((s = \sigma + j\omega)\)
\(v\) velocity
\(y\) state vector of controlled 1-port
\(Y(s)\) admittance
\(z\) state vector of environment
\(Z(s)\) impedance
\(\varepsilon\) generalized effort (force)
\(\phi\) generalized flow (velocity)
states" and "marginal stability." In Section 5, a solution to the first problem and the new definition of SPR are presented, and in Section 6, conclusions are drawn.

2. Formulation of the Problem

Consider the coupled system illustrated in Figure 1(a) consisting of a controlled 1-port (e.g., the robot) connected via a common flow junction (in bond graph terminology, a 1-junction [16]) to a passive environment. This block diagram may be rearranged to yield the more compact version shown in Figure 1(b), which clearly separates the feedback loop comprising the admittance, $Y(s)$, and the impedance, $Ze(s)$, from the feedforward contribution of any state-independent effort or flow sources (e.g., gravitational pull) and of initial conditions. For the purposes of this paper, the following constraints are placed on the environment:

i. $Ze(s)$ is PR, containing no pole-zero cancellations.
ii. The effort, $\varepsilon(t)$, due to state-independent effort and flow sources within the environment, and to the initial conditions of environmental states, is bounded.

Otherwise, the environment is arbitrary. Because coupled stability subsumes the isolated stability of the 1-port [6], it is clear that the flow, $\phi(t)$, will be bounded. From this point forward, the inputs to the coupled system will be taken as $\varepsilon_i$ and $\phi_i$, unless otherwise stated.

The coupled system is said to be *internally stable* if each of the elements of the following transfer function matrix is exponentially stable [15]:

$$
\begin{bmatrix}
\phi_z(s) \\
\varepsilon_y(s)
\end{bmatrix} =
\begin{bmatrix}
[1 + Y(s)Ze(s)]^{-1} & -[1 + Y(s)Ze(s)]^{-1}Y(s) \\
[1 + Ze(s)Y(s)]^{-1}Ze(s) & -[1 + Ze(s)Y(s)]^{-1}
\end{bmatrix}
\begin{bmatrix}
\phi_i(s) \\
\varepsilon_i(s)
\end{bmatrix}
$$

(1)

It is now possible to state the following problem, whose solution is the principal objective of this paper:

**Problem 1.** What conditions must $Y(s) = D + C(sI - A)^{-1}B$ satisfy such that, for any admissible environment, the coupled system is internally stable?
The only *a priori* information available concerning $Y(s)$ is that it is exponentially stable (coupled stability subsumes isolated stability), but this is immediately useful. Maciejowski [15] shows that, under this condition, the coupled system is internally stable if and only if the lower left term, $[1+Z_e(s)Y(s)]^{-1}Z_e(s)$, is exponentially stable. In other words, the exponential stability of this transfer function implies that of the other three. While the solution to Problem 1 may now seem to be easily obtainable with the use of Nyquist arguments, it is not yet, due to a subtlety that is the subject of Section 4.

In robotics, velocities are often a greater concern than forces when assessing stability. For instance, if a short pulse of force is applied to the tip of a robot, and the velocity of some joint of the robot does not thereafter decay to zero, then the robot is not considered to be stable. However, if the converse experiment is performed and results are found; i.e., if a short pulse of velocity (a displacement) is applied to the tip of the robot, and the force at some joint does not decay to zero, the robot is not necessarily deemed unstable. In fact, this is a behavior — stiffness — that is intentionally implemented in most robots. This matter will be further addressed in Section 4; however, these examples immediately suggest that it may sometimes be useful to consider the asymptotic stability of only the port flow, $\phi_z$, and not of the port effort, $\varepsilon_y$. Therefore, Problem 2 is posed:

**Problem 2.** What conditions must $Y(s)$ satisfy such that, for any admissible environment, the port flow, $\phi_z$, is exponentially stable?

For this problem, it is clearly sufficient to find the conditions under which the transfer function $[1+Y(s)Z_e(s)]^{-1}$ is exponentially stable. This is done in the next section.

### 3. Solution of Problem 2

A sufficient condition for Problem 2 is easily found: it is simply that $Y(s)$ be SPR. Actually, a variety of definitions of SPR have been given in the literature, and any of these will be valid here. They include:

1. $Y(s)$ is exponentially stable, and $\Re\{Y(j\omega)\} > \delta \quad \forall \omega \in \mathbb{R}$ and some $\delta > 0$ [8].
ii. \( Y(s-\epsilon) \) is PR for some \( \epsilon > 0 \) (equivalently, \( Y(s) \) is exponentially stable, and \( \Re\{Y(j\omega)\} > 0 \ \forall \omega \in \mathbb{R} \) and does not approach 0 any faster than \( \omega^{-2} \) as \( \omega \to \infty \)) [17].

iii. \( Y(s) \) is exponentially stable, and \( \Re\{Y(j\omega)\} > 0 \ \forall \omega \in \mathbb{R} \) [14].

All of these criteria guarantee that \(-90^\circ < \angle Y(j\omega) < 90^\circ \ \forall \omega \in \mathbb{R} \). Because \(-90^\circ \leq \angle Z_e(j\omega) \leq 90^\circ \ \forall \omega \in \mathbb{R} \), it follows that \(-180^\circ < \angle Y(j\omega)Z_e(j\omega) < 180^\circ \ \forall \omega \in \mathbb{R} \). The Nyquist Theorem, therefore, ensures that the poles of \([1 + Y(s)Z_e(s)]^{-1}\) lie in the open left half plane [7]. \( \phi_2 \), therefore, is exponentially stable.

It is worth noting that the condition above is also sufficient for Problem 1. If \( Y(s) \) and \( Z_e(s) \) are expressed in terms of their numerator and denominator polynomials, the exponentially stable transfer function \([1 + Y(s)Z_e(s)]^{-1}\) may be written as:

\[
[1 + Y(s)Z_e(s)]^{-1} = \frac{D_y(s)D_z(s)}{D_y(s)D_z(s) + N_y(s)N_z(s)}.
\] (2)

The transfer function \([1+Z_e(s)Y(s)]^{-1}Z_e(s)\) may be written as:

\[
[1 + Z_e(s)Y(s)]^{-1}Z_e(s) = \frac{D_y(s)N_z(s)}{D_y(s)D_z(s) + N_y(s)N_z(s)}.
\] (3)

These transfer functions will have the same poles (thus, the coupled system will be internally stable) so long as there are no pole-zero cancellations in the product \( Y(s)Z_e(s) \). If such a pole-zero cancellation occurs in the right half plane, the coupled system will not necessarily be internally stable. However, if \( Y(s) \) is SPR according to any of the definitions given above, it cannot contain either poles or zeros in the closed right half plane; accordingly, the coupled system will be internally stable.

None of the existing definitions of SPR, however, provides a necessary condition for Problem 2. This can be illustrated with a simple example taken from robotics. Figure 2 is a 1-port, linear version of an impedance-controlled robot [10]. \( M \) represents the mass of the robot arm, while \( K \) and \( B \) represent the stiffness and damping of the feedback controller, respectively. The massless cart represents the computer-commanded "virtual position." If the virtual position does not vary with time, the cart acts as though it is rigidly connected to ground. The admittance of this robot is:
\[ Y_1(s) = \frac{s}{M s^2 + B s + K}. \] (4)

Obviously, \( \text{Re}\{Y_1(j\omega)\} = 0 \) for \( \omega = 0 \), violating all versions of SPR; yet, when this system is coupled to any admissible environment, its endpoint velocity is exponentially stable. This may be demonstrated as follows. The poles of the transfer function \([1 + Y(s)Z_e(s)]^{-1}\) are the roots of:

\[ M s^2 + (R + Z_e(s))s + K = 0. \] (5)

Because both \( Y(s) \) and \( Z_e(s) \) are PR, the poles cannot have positive real parts \([7]\). Suppose, however, that \( s = j\omega \) is a solution. Then, the following equation must be satisfied for some \( \omega \):

\[ [K - M\omega^2 - \omega \text{Im}\{Z_e(j\omega)\}] + j\omega[B + \text{Re}\{Z_e(j\omega)\}] = 0. \] (6)

Because \( \text{Re}\{Z_e(j\omega)\} \geq 0 \), the only possible solution is \( \omega = 0 \), which leads to the following condition:

\[ K = \lim_{\omega \to 0} \left\{ \omega \text{Im}\{Z_e(j\omega)\} \right\}. \] (7)

One consequence of Eqn. 7 is that \( \text{Im}\{Z_e(j\omega)\} \to \infty \) as \( \omega \to 0 \); thus, \( Z_e(s) \) includes a pole at the origin. Because \( Z_e(s) \) is PR, the pole must be simple and its residue must be a positive real number \([4]\); thus, the limit on the right hand side of Eqn. 7 must be zero. This produces a conflict and demonstrates that the endpoint velocity is asymptotically stable as claimed.

The complete solution to Problem 2 is provided by the following theorem:

**Theorem 1.** Consider the coupled system shown in Figure 1(b), with \( Z_e(s) \) PR, but otherwise arbitrary, and \( \epsilon(t) \) bounded. Necessary and sufficient conditions for the port flow, \( \phi_e(t) \), to be exponentially stable are:

1. \( Y(s) \) is exponentially stable.
2. \( \text{Re}\{Y(j\omega)\} > 0 \ \forall \omega \in \mathbb{R} \) unless \( s = j\omega \) is a simple zero of \( Y(s) \).

**Proof**

- Condition 1 is necessary because coupled stability subsumes isolated stability.
• If $\text{Re}\{Y(j\omega)\}<0$ for any $\omega$, a destabilizing $Z_e(s)$ which is PR can always be found according to the proof in [7].

• Suppose $\text{Re}\{Y(j\omega)\}=0$ for some $\omega$. Three cases can be considered:
  i. $\text{Im}\{Y(j\omega)\}=0$. $Y(s)$ contains a zero at $s=j\omega$. If the zero is not simple, $Y(s)$ is not PR, and some destabilizing $Z_e(s)$ exists [7]. If the zero is simple, $Y(s)$ may be written as:

$$Y(s) = \frac{N_y(s)}{D_y(s)} = \frac{(s-j\omega)N_y^*(s)}{D_y(s)}.$$  \hspace{1cm} (8)

It is clear that no admissible environment exists that will result in a coupled system with exponentially unstable poles. Assume that $Z_e(s)$ exists such that $Z_e(j\omega)Y(j\omega) = -1$; this would result in a coupled system with a pole at $s=j\omega$. Such a $Z_e(s)$ must include a pole at $s=j\omega$:

$$Z_e(s) = \frac{N_z(s)}{D_z(s)} = \frac{N_z(s)}{(s-j\omega)D_z^*(s)},$$  \hspace{1cm} (9)

and:

$$Y(j\omega)Z_e(j\omega) = \frac{N_y^*(j\omega)N_z(j\omega)}{D_y(j\omega)D_z^*(j\omega)} = -1.$$  \hspace{1cm} (10)

However, because both $Y(s)$ and $Z_e(s)$ are PR, it is necessary that $\lim_{s \to \infty} Y(s)/(s-j\omega) \geq 0$, and $\lim_{s \to \infty} (s-j\omega)Z_e(s) \geq 0$ [4]; thus, Eqn. 10 cannot be true and no destabilizing $Z_e(s)$ exists.

ii. $-\infty < \text{Im}\{Y(j\omega)\} < 0$ or $0 < \text{Im}\{Y(j\omega)\} < \infty$. In this case, $\angle Y(j\omega) = \pm 90^\circ$, and some $Z_e(s)$ can clearly be found such that $Y(j\omega)Z_e(j\omega) = -1$, indicating that $s=j\omega$ is a pole of the coupled system.

iii. $\text{Im}\{Y(j\omega)\} \to \pm \infty$. In this case, $Y(s)$ must have a pole at $s=j\omega$, violating condition 1. □

The system shown in Figure 2 satisfies the conditions of Theorem 1. Another system which satisfies these conditions is shown in Figure 3(a). Its admittance is:
\[ Y_2(s) = \frac{Ms^2 + K}{MBs^2 + MKs + BK} \]

It is easy to verify that \( \text{Re}\{Y_2(j\omega)\} > 0 \) for all \( \omega \) except \( \omega = \pm \sqrt{K/M} \). Although the endpoint velocity will be exponentially stable when this system is coupled to any passive environment, the velocity of the mass will not be, necessarily. When the system is coupled to the environment shown in Figure 3(b), it is clear that a resonant mode exists in which the two masses oscillate 180° out of phase (this mode is unobservable from the endpoint velocity). This example makes it clear that the conditions of Theorem 1 do not provide a solution to Problem 1.

4. Excess States Versus Marginal Stability

An essential prelude to the solution of Problem 1 is to introduce the concept of an "excess state," and to distinguish it from the concept of "marginal stability." The latter term is the more common, referring to the behavior of a system with a pole or poles lying on the imaginary axis (but no poles in the open right half plane). The system in Figure 4(a), for instance, is recognized as marginally stable due to the undamped rigid body motion.

An excess state occurs when knowledge of the state variable associated with one independent (integral causality) storage provides instantaneous knowledge of the state variable associated with another independent storage (Andry and Rosenberg [3] provide examples of systems with excess states in their discussion of state space dimension, but analyze only examples in which derivative causality accounts for the excess states). As an example, the system in Figure 4(b) contains an excess state. The two springs can be replaced with a single spring of stiffness \( K_1 + K_2 \) without affecting the equations of motion. To elaborate this point, third order state equations can be written as follows:

\[
\begin{bmatrix}
\frac{d}{dt} v \\
\begin{bmatrix} f_1 \\
f_2 \\
\end{bmatrix}
\end{bmatrix} = \begin{bmatrix}
-B/M & -1/M & -1/M \\
K_1 & 0 & 0 \\
K_2 & 0 & 0
\end{bmatrix} \begin{bmatrix}
v \\
f_1 \\
f_2 \\
\end{bmatrix},
\]

(11)
where \( v \) is the velocity of the mass, \( f_1 \) is the force in spring 1 (positive in tension), and \( f_2 \) is the force in spring 2 (positive in compression). The determinant of the \( A \)-matrix is zero, indicating that \( s=0 \) is an eigenvalue. Therefore, one might conclude that this system is marginally stable, although such behavior is not nearly so obvious as it is for the system in Figure 4(a). A better conclusion might be that the system includes an excess state: knowledge of the force in spring 1 provides instantaneous knowledge of its length, and therefore of the length and force of spring 2. The excess state can be removed by defining a new state as follows:

\[
F = f_1 + f_2. \tag{12}
\]

The reduced state equations are:

\[
\begin{bmatrix}
\frac{dv}{dt} \\
\frac{dF}{dt}
\end{bmatrix} =
\begin{bmatrix}
-B/M & -1/M \\
K_1+K_2 & 0
\end{bmatrix}
\begin{bmatrix}
v \\
F
\end{bmatrix}. \tag{13}
\]

The pole at the origin has been removed and the remaining poles are exponentially stable.

It is important to ask, however, what information, if any, has been lost in reducing the state equations. One approach to this question is to consider the space of possible outputs. From Eqn. 11, any output that is a function of \( v, f_1, \) and \( f_2 \) can be produced. Obviously, the same may be said of Eqn. 13 if \( f_1 \) and \( f_2 \) are among the possible outputs. Unfortunately, this is not the case, in general. Suppose that, when the system is at rest, a preload of \( f_1=f_0 \) and \( f_2=-f_0 \) exists in the two springs. Then it is straightforward to show that the following output equation holds:

\[
\begin{bmatrix}
f_1-f_0 \\
f_2+f_0
\end{bmatrix} =
\begin{bmatrix}
0 & K_1/(K_1+K_2) \\
0 & K_2/(K_1+K_2)
\end{bmatrix}
\begin{bmatrix}
v \\
F
\end{bmatrix}. \tag{14}
\]

Thus, \( f_1 \) and \( f_2 \) can be determined only to within the preload value (the existence of a non-decaying preload accounts for the pole at \( s=0 \) in the full state equations). To justify the use of Eqn. 13, it is necessary either to know that \( f_0=0 \), or to limit one's interest to outputs that can be expressed in terms of the relative forces. In the context of robotics, either of these may prove a valid reason for using the reduced equations.
It is interesting to note that virtually identical arguments could have been made for the system in Figure 4(a). A two-dimensional state vector can be found in the center of mass frame. However, this requires that the velocity of the center of mass be either zero or ignorable, neither of which is likely to be acceptable in robotics. This example underscores the fundamental difference in the way that forces and velocities are treated in assessing the stability of mechanical systems. A full explanation of the appropriate roles of forces and velocities is beyond the scope of this paper; however, it is a goal of this paper to exploit the distinction between these roles in a useful definition of SPR. All previous definitions of SPR have been entirely mathematical in their nature, and consequently have been completely symmetric with respect to efforts and flows.

A final issue with regard to excess states is the treatment of inputs. Consider, for instance, the system illustrated in Figure 4(c) (which may be thought of as a coupled system consisting of the 1-port in Figure 2 and a sping-like environment). State equations similar to those in Eqn. 11 may be derived; however, the equations for $df_1/dt$ and $df_2/dt$ are no longer dependent:

$$
\frac{d}{dt} \begin{bmatrix} v \\ f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -B/M & -1/M & 1/M \\ K_1 & 0 & 0 \\ K_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} B/M \\ -K_1 \\ 0 \end{bmatrix} v_i.
$$

(15)

Nonetheless, a new state may be defined as in Eqn. 12 and the state equations may be reduced. Again, $f_1$ and $f_2$ may be found as outputs to within a constant preload, $f_0$; however, it is necessary to have available as an input the position of the cart, $x_i(t)$, as well as the velocity of the cart, $v_i(t)$:

$$
\frac{d}{dt} \begin{bmatrix} v \\ F \\ f_1-f_0 \\ f_2+f_0 \end{bmatrix} = \begin{bmatrix} -B/M & -1/M \\ K_1+K_2 & 0 \end{bmatrix} \begin{bmatrix} v \\ F \end{bmatrix} + \begin{bmatrix} 0 & B/M \\ 0 & -K_1 \end{bmatrix} \begin{bmatrix} x_i \\ v_i \end{bmatrix},
$$

(16)

It may well be argued that it is improper to consider $x_i(t)$ an input — that it should be a state, and that consequently, the system will contain a pole at $s=0$. Such a state equation, however, would represent a kinematic relationship and not an energetic behavior; and as such, would not contribute to the dynamics or stability of the system.
In the development above, the preload, $f_0$, and the cart position, $x_i(t)$, are considered independent quantities; however, from a physical standpoint, it is clear that they need not be. It is possible to choose the zero position of the cart such that, when the system is in equilibrium ($x_i=0$, $v_i=0$, $v=0$, and $F=0$), the preload is zero. With this selection of zero position, explicit output equations can be written for $f_1$ and $f_2$:

$$
\begin{bmatrix}
  f_1 \\
  f_2 
\end{bmatrix} =
\begin{bmatrix}
  0 & K_1/(K_1+K_2) \\
  0 & K_2/(K_1+K_2)
\end{bmatrix}
\begin{bmatrix}
  v \\
  F
\end{bmatrix} +
\begin{bmatrix}
  -K_1K_2/(K_1+K_2) \\
  K_1K_2/(K_1+K_2)
\end{bmatrix}
\begin{bmatrix}
  x_i \\
  v_i
\end{bmatrix}
$$

(17)

Thus, with the addition of an appropriately selected input, $x_i(t)$, the reduced state equations provide all the information of the full state equations.

5. Solution of Problem 1

The solution to Problem 1 is based upon the two premises stated below, which are consequences of the discussion in Section 4. Both are derived from concepts in physical systems theory rather than concepts in control theory.

**Premise 1.** For the purpose of assessing coupled stability, robots should always be modeled with the admittance causality (effort input, flow output). This premise will be called the "principal asymmetry," because it provides the basic distinction between forces and velocities that was motivated in Section 4. By modeling robots as admittances, it is assured that a zero at the origin is indicative of an equivalent stiffness connecting the endpoint (port) to ground, not of an undamped rigid body mode. A more general reason is that the admittance causality enforces a zero impedance boundary condition on the isolated robot, which is intuitively appealing. Impedance causality, on the other hand, enforces an infinite impedance boundary condition, which effectively requires that the "isolated" robot be coupled to a rigid wall. This is non-intuitive, and can result in important dynamic effects being obscured. Finally, most robots are primarily inertial, and the preferred (integral) causality of inertias is the admittance causality.

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$^2$In Hogan's works on "impedance control," the principal asymmetry consists of treating the robot as an impedance and the environment as an admittance. One motivation for this choice is that many environments are best modeled as admittances [10].
Premise 2. If \( \det(A_{\text{coupled}})=0 \), and a state can be eliminated with the loss of preload information only, then that state is considered to be an "ignorable excess state."

The solution to Problem 1 is provided by the following theorem:

**Theorem 2.** Consider the coupled system shown in Figure 1(b), with \( Z_e(s) \) PR, but otherwise arbitrary, and \( \varepsilon_i(t) \) bounded. Necessary and sufficient conditions for the system to be internally stable are:

1. \( Y(s) \) is exponentially stable.
2. \( \text{Re}\{Y(j\omega)\} > 0 \ \forall \omega \in \mathbb{R} \) unless \( s=0 \) is a simple zero of \( Y(s) \).

An admittance that satisfies the conditions of this theorem is said to be SPR.

**Proof**

It is necessary only to show that the poles of \( [1+Z_e(s)Y(s)]^{-1}Z_e(s) \) are exponentially stable for any \( Y(s) \) that satisfies the conditions of Theorem 1, unless \( s=j\omega \), \( \omega \neq 0 \) is a zero of \( Y(s) \).

- If \( Y(s) \) contains no zeros on the imaginary axis, then the poles of \( [1+Z_e(s)Y(s)]^{-1}Z_e(s) \) are exponentially stable according to the arguments given in Section 3.
- Suppose \( s=j\omega \) is a zero of \( Y(s) \), and a pole of some \( Z_e(s) \). The term \( [1+Z_e(s)Y(s)]^{-1}Z_e(s) \) may be rewritten using Equations 8 and 9:

\[
[1+Z_e(s)Y(s)]^{-1}Z_e(s) = \frac{N_z(s)D_y(s)}{(s-j\omega)[D_y(s)D_z'(s) + N'_y(s)N_z(s)]}. \tag{18}
\]

Therefore, \( s=j\omega \) is a pole of the coupled system. If \( \omega \neq 0 \), it follows that the coupled system is not internally stable.

- If \( \omega=0 \), the pole lies at the origin, and it is possible that the coupled system contains an ignorable excess state. To demonstrate that this is the case in general, consider the coupled system comprised of an admittance, \( Y(s) \), with a zero at the origin, and a PR impedance, \( Z_e(s) \), with a pole at the origin:

\[
Y(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0},
\]
\[ Y(s) = b_n + \frac{(b_{n-1} - b_n a_{n-1})s^{n-1} + (b_{n-2} - b_n a_{n-2})s^{n-2} + \cdots + (b_1 - b_n a_1)s - b_n a_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}. \] (19)

\[ Z_e(s) = \frac{\beta_m s^m + \beta_{m-1}s^{m-1} + \cdots + \beta_1 s + \beta_0}{s^m + \alpha_{m-1}s^{m-1} + \cdots + \alpha_1 s}, \]

\[ Z_e(s) = \beta_m + \frac{(\beta_{m-1} - \beta_m a_{m-1})s^{m-1} + (\beta_{m-2} - \beta_m a_{m-2})s^{m-2} + \cdots + (\beta_1 - \beta_m a_1)s + \beta_0}{s^m + \alpha_{m-1}s^{m-1} + \cdots + \alpha_1 s}. \] (20)

To reveal an excess state, state equations for the coupled system are required. State equations for the 1-port and the environment can be found in observable canonical form:

\[
\frac{dy}{dt} = \begin{bmatrix} 0 & 0 & \cdots & -a_0 \\ -a_1 \\ \vdots \\ -a_{n-1} \end{bmatrix} y + \begin{bmatrix} -b_n a_0 \\ b_1 - b_n a_1 \\ \vdots \\ b_{n-1} - b_n a_{n-1} \end{bmatrix} \varepsilon_y
\]

\[
\phi_y = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} y + b_n \varepsilon_y. \] (21)

\[
\frac{dz}{dt} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -\alpha_1 \\ \vdots \\ -\alpha_{m-1} \end{bmatrix} z + \begin{bmatrix} \beta_0 \\ \beta_1 - \beta_m a_1 \\ \vdots \\ \beta_{m-1} - \beta_m a_{m-1} \end{bmatrix} \phi_z
\]

\[
\varepsilon_z = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} z + \beta_m \phi_z. \] (22)

The state \( z_1 \) can be given a physical interpretation. In his seminal work on network synthesis, Brune showed that a pole at \( s=0 \) can be extracted from a positive real function via partial fraction expansion, leaving a positive real remainder \([4]\). Therefore, \( Z_e(s) \) can be written:

\[
Z_e(s) = \frac{\beta_0}{\alpha_1 s} + Z_e'(s), \] (23)
where \( Z_e(s) \) is positive real. A corresponding mechanical realization of \( Z_e(s) \) is shown in Figure 5. The force in the spring is \( \beta_0/\alpha_1 \) times the port displacement. The state \( z_1 \), therefore, is just the constant \( \alpha_1 \) times the force in the spring (within a constant offset). If it can be shown that \( z_1 \) is an excess state, it is justifiably ignorable.

State equations for the coupled system shown in Figure 1(b) can be found by the straightforward, though tedious, elimination of \( \varepsilon_y \) and \( \phi_i \). The result is:

Equation 24 is shown on page 16.

The state equations for \( y_1 \) and \( z_1 \) can be linearly combined to produce:

\[
\frac{dz_1}{dt} + \frac{\beta_0}{a_0} \frac{dy_1}{dt} = \beta_0 \phi_i(t).
\] (25)

This equation can be integrated to solve for \( z_1 \) in terms of \( y_1 \), \( \phi_i \), and initial conditions:

\[
z_1(t) = -\frac{\beta_0}{a_0} y_1(t) + \beta_0 \left[ \int_{t_o}^{t} \phi_i(t) dt + \frac{z_1(t_o)}{\beta_0} + \frac{y_1(t_o)}{a_0} \right].
\] (26)

The term \( \left[ \int_{t_o}^{t} \phi_i dt + z_1(t)/\beta_0 + y_1(t)/a_0 \right] \) can be considered an input which represents the effects of feedforward displacements and initial conditions (preload). If the zero position of the feedforward displacement is selected such that:

\[
\int_{-\infty}^{t_o} \phi_i dt = \frac{z_1(t_o)}{\beta_0} + \frac{y_1(t_o)}{a_0},
\] (27)

then \( z_1 \) can be determined instantaneously from a knowledge of \( y_1 \) and the feedforward displacement. Therefore, \( z_1 \) is an ignorable excess state. A reduced state vector \( [y \ z_r] \) can be defined, and state equations written:

Equation 28 is shown on page 16.
In addition to removing the excess state, the pole at the origin has been removed. This can be verified by examining the $A$-matrix of the reduced coupled system. Due to the placement of the identity submatrices, it is clear that the determinant of this matrix will be zero only if the determinant of the $3 \times 3$ matrix formed by the intersection of the first, second, and $n+1^{st}$ rows and columns is zero:

$$
\begin{vmatrix}
0 & a_0 & a_0 b_n \\
1 & a_1 + b_1 \beta_m & \frac{a_1 b_n - b_1}{1 + b_n \beta_m} \\
1 & a_1 \beta_m \cdot \frac{1}{1 + b_n \beta_m} & \frac{b_1 \beta_0}{1 + b_n \beta_m}
\end{vmatrix} = 0. \tag{29}
$$

Expanding and multiplying by $(1+b_n\beta_m)^2$, which must be positive, gives:

$$-b_1 \beta_0 - \alpha_1 a_0 - a_0 b_n \alpha_1 \beta_m - b_1 b_n \beta_0 \beta_m = 0. \tag{30}$$

Because neither $Y(s)$ nor $Z_e(s)$ contains any poles or zeros in the open right half plane, all coefficients must be greater than or equal to zero; therefore, each term in Eqn. 30 must be zero. It is possible that either $b_n$ or $\beta_m$ is zero; however, neither $a_0$ nor $\beta_0$ can be zero, because both $Y(s)$ and $Z_e(s)$ are minimal; nor can $b_1 = 0$ or $\alpha_1 = 0$, because $Y(s)$ has at most a simple zero at $s = 0$, and $Z_e(s)$ has at most a simple pole at $s = 0$. Therefore, the reduced system does not have a pole at the origin.

Referring now to Eqn. 18, the poles of the coupled system will be the roots of $D_y(s)D_z(s) + N'_y(s)N_z(s)$. According to the Nyquist arguments presented in Section 3, these roots will all be exponentially stable. Thus, the coupled system is internally stable. ■

6. Conclusions

In this paper, a new definition of SPR (provided by Theorem 2) was developed for use in assessing the stability of coupled mechanical systems. A linear, time-invariant 1-port that is SPR according to this definition will be exponentially stable.
when coupled to any linear, time-invariant environment that has a PR impedance (more precisely, the coupled system will be internally stable). This definition will be particularly useful in providing a robustness constraint with which to guide the design of feedback controllers for robots.

Future work should include an extension of this definition to multi-port admittances, and an extension of the stability proof to include a class of nonlinear, passive environments. A similar definition would prove very useful in the context of telemanipulation, where the "environment" consists of two distinct multi-ports, the operator and the true environment [5].

References


Figure 1. (a) Block diagram description of the coupled system. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$, $D \in \mathbb{R}$, $L \in \mathbb{R}^{n \times p}$; $A_e \in \mathbb{R}^{m \times m}$, $B_e \in \mathbb{R}^{m \times 1}$, $C_e \in \mathbb{R}^{1 \times m}$, $D_e \in \mathbb{R}$, $L_e \in \mathbb{R}^{m \times d}$. $(A,B,C,D)$ and $(A_e,B_e,C_e,D_e)$ are minimal realizations of $Y(s)$ and $Z_e(s)$, respectively. $y$ is the state vector of the controlled 1-port, $z$ is the state vector of the environment; $y_0$ and $z_0$ are initial conditions of the states, and $u_y$ and $u_z$ are exogenous inputs. (b) Reduced block diagram of the coupled system.
Figure 2. Translational 1-port representation of an impedance-controlled robot.
Figure 3. (a) Example of a 1-port that satisfies the conditions of Theorem 1. The port velocity of this system will remain asymptotically stable when the system is coupled to any admissible passive environment. (b) Example of an admissible passive environment which, when coupled to the 1-port shown in (a), will result in an undamped mode of oscillation.
Figure 4. (a) Example of a marginally stable translational mechanical system. (b) Example of a mechanical system with an excess state. Note that the bond graph does not indicate derivative causality on any storage; yet, the two capacitive storages can clearly be combined. (c) System with the same internal dynamics as (b), but with the left wall replaced by a flow source (cart). The presence of an excess state is not obvious from the bond graph.
Figure 5. Partial realization of a PR impedance with a pole at the origin.