Coordinate Transformations and Logical Operations for Minimizing Conservativeness in Coupled Stability Criteria

J. E. Colgate
Department of Mechanical Engineering, Northwestern University, Evanston, IL 60208-3111

Often it is desirable to guarantee that a manipulator will remain stable when contacting any member of some set of environments. Coupled stability criteria based on passivity may be used to provide such a guarantee, but may be arbitrarily conservative depending on the environment set. In this paper, two techniques for reducing conservativeness are introduced. The first is based on a canonical coordinate transformation which enables an environment set viewed in the frequency domain to be conformally mapped to the interior of the unit circle. A stability criterion is then derived via the small gain theorem. The second technique uses logical combinations of such criteria to reduce conservativeness further. Both techniques are illustrated with nontrivial examples.

I Introduction

This paper addresses the generation of "coupled stability criteria"—conditions which guarantee the stability of a manipulator in contact with ("coupled to") any member of some well-defined class of physical environments. This work stems from previous results in which the class of environments was assumed to comprise all passive systems (Colgate and Hogan, 1988).

The notion of passivity has a rich history in the analysis of system stability. Passivity plays a central role in the theories of absolute stability put forth by Popov (1973) and others (Narendra and Taylor, 1973), and in stability proofs for various adaptive control schemes (Landau, 1979). More recently, passivity was applied to the stability analysis of robots (Colgate, 1989) and teleoperators (Anderson and Spong, 1989); and is now widely employed as a design constraint in the development of manipulator controllers (Cotsaftis, 1992; Newman, 1992; Niemeyer and Slotine, 1991). Recently, however, it has been argued that passivity is an unattainable or even undesirable goal in the face of real implementation effects, such as sampled-data, sensor and actuator dynamics, etc. This has motivated the investigation of "nearly passive" robots (Chapel and Su, 1992), as well as the definition of a "distance to passivity" (Andriot and Fournier, 1992).

In this paper, a substantially more general approach to the generation of nonconservative coupled stability criteria will be presented. This approach is developed in three steps: Step 1 employs the small gain theorem to establish simple but very conservative conditions; Step 2 employs a canonical coordinate transformation to develop less conservative conditions; and Step 3 employs logical combinations of conditions established in Step 2 to reduce conservativeness even further. Each of these conditions applies to linear time invariant manipulator and environment descriptions only, although extensions to nonlinear systems will be mentioned.

II Coupled Stability Via the Small Gain Theorem

When a manipulator contacts[1] a physical environment, a bilateral interaction is created that can be modeled as a feedback loop (Hogan, 1988, Kazerooni et al., 1986). Bilateral interaction is most often described in terms of dynamic operators called impedances and admittances which map

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[1] The terms "contact," which implies unilateral constraint, and "couple," which implies bilateral constraint, will be used interchangeably in this paper (except in Example 3). This is because the act of making and breaking contact with an environment cannot, for a stable manipulator with continuous time control, be the source of instability (Colgate, 1988). Note, however, that under sampled data control, the coupled stability of a manipulator does not necessarily imply contact stability (Colgate et al., 1993).

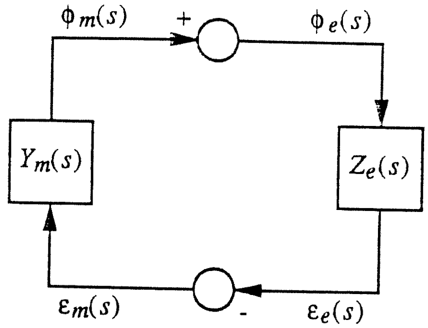

Fig. 1

flows (ϕ) to efforts (ϵ) and efforts to flows, respectively. "Flow" refers to a generalized vector of velocities and angular velocities, and "effort" refers to a generalized vector of forces and torques. The inner product of effort and flow is the instantaneous power input to the associated physical system. In terms of a manipulator's admittance, $Y_m(s)$, and an environment's impedance, $Z_e(s)$, coupling creates the feedback structure depicted in Fig. 1. The environment impedance may generally be identified as the member of some set ($\mathbb{S}_e^n$), for instance, the set of positive real functions. Typically, this set will be defined through inequality constraints imposed in the frequency domain, as will become evident in the examples.

The small gain theorem (Desoer and Vidyasagar, 1975) guarantees the stability of the feedback system in Fig. 1 if $\|Z_e(s)Y_m(s)\|_\infty < 1$, where $Z_e(s)$ and $Y_m(s)$ are assumed stable. Stability, here, and throughout this paper, means that all poles are contained within the open left half plane (BIBO stability). Because the infinity norm is submultiplicative, stability of the feedback system is guaranteed if:

$$\|Z_e(s)\|_\infty \le \gamma \quad \text{and} \quad \|Y_m(s)\|_\infty < 1/\gamma \tag{1}$$

or, more generally, if:

$$\|W_1^{-1} Z_e W_2^{-1}\|_\infty \le 1 \quad \text{and} \quad \|W_2 Y_m W_1\|_\infty < 1 \tag{2}$$

where the argument s has been dropped for brevity, and where $W_1(s)$ and $W_2(s)$ are stable, minimum phase transfer function matrices. This condition is identical in form to a standard condition for robustness to multivariable multiplicative uncertainties (Maciejowski, 1989).

In the 1-port (SISO) case, W_2 may be eliminated, and a simple graphical interpretation is available[2]. If at every frequency $0 \le \omega < \infty$, $Z(j\omega)$ lies within a disk centered at the origin with a radius $|W_1(j\omega)|$, the coupled system is guaranteed stable if $|Y(j\omega)|$ lies strictly within a disk of radius $|W_1^{-1}(j\omega)|$, also centered at the origin. This is an obvious consequence of the Nyquist criterion. It is evident, however, that this condiion may be exceedingly conservative if the disk covering the environment set is substantially larger than the set.

Example 1. Consider the set of environments consisting of damped springs having stiffness k less than some maximum, $k_{\max}$, and damping ϵ. Then $Z_e(s) = k/s + \epsilon$, $0 \le k \le k_{\max}$, $\epsilon > 0$. A good choice for the weighting function would be $W_1(s) = k_{\max}/s + \epsilon$. It is then clear that $\|W_1^{-1}Z_e\|_\infty = \left\|\frac{k + \epsilon s}{k_{\max} + \epsilon s}\right\|_\infty \le 1$, and that coupled stability can be guaranteed by $\|Y_m(k_{\max}/s + \epsilon)\|_\infty < 1$. This result, however, is rather conservative because the disk of radius $|W_1(j\omega)| = \sqrt{(k_{\max}/\omega)^2 + \epsilon^2}$ covers much more than the actual environment set, which is described by a line from $(\epsilon, 0)$ to $(\epsilon, -jk_{\max}/\omega)$. For instance, a spring of stiffness $-k_{\max}$ is covered by the disk. Thus, when satisfied, the criterion ensures the stability of Y_m coupled to a negative stiffness spring, which is unnecessary and highly conservative. ■

This example illustrates a very important point—the small gain theorem takes no phase information into account, and therefore assumes complete phase uncertainty in the environment set (thus, the set is disk-shaped). Admittances and impedances, however, are generally not completely phase uncertain. For instance, it is well known that the phase of a passive environment's impedance must lie in the interval $[-\pi/2, +\pi/2]$.

Often, however, there is a way around this difficulty.

[2] Throughout this paper, graphical interpretations should be understood to apply to the 1-port case only, while all mathematical results apply to the n-port case unless otherwise stated.

Nomenclature

- a = scalar or vector
- A = matrix
- A^* = conjugate transpose of A
- $A \ge 0$ = A is positive semi-definite
- $T(A)$ = linear fractional transformation of A
- $\bar{\sigma}(A)$ = maximum singular value of A
- $\|A(s)\|_\infty$ = $\sup_\omega (\bar{\sigma}(A(j\omega)))$
- Z = impedance matrix
- Y = admittance matrix
- ϕ = flow vector
- ϵ = effort vector
- $\mathbb{C}$ = set of complex numbers
- $\mathbb{C}^n$ = set of complex-valued $n \times n$ matrices
- $\mathbb{R}_+$ = set of non-negative real numbers
- $\mathbb{C}^n\mathbb{R}_+$ = set of matrix-scalar pairs $(M, \omega) \mid M \in \mathbb{C}^{n\times n}$, $\omega \in \mathbb{R}_+$ ($\mathbb{C}^1\mathbb{R}_+$ may be visualized as the "extruded" complex plane, as in Fig. 5)

Generic sets:

- $\mathfrak{S}^n$ = $\mathfrak{S}^n \subset \mathbb{C}^n$
- $\mathbb{S}^n$ = $\mathbb{S}^n \subset \mathbb{C}^n\mathbb{R}_+$

Special sets:

- $\mathfrak{D}_1^n$ = $\mathfrak{D}_1^n \subset \mathbb{C}^n \mid M \in \mathfrak{D}_1^n$, $\bar{\sigma}(M) \le 1$ ($\mathfrak{D}_1^1$ is the "unit disk")
- $\mathbb{D}_1^n$ = $\mathbb{D}_1^n \subset \mathbb{C}^n\mathbb{R}_+ \mid \bar{\sigma}(M) \le 1$ ($\mathbb{D}_1^1$ is the "unit cylinder")
- $\mathfrak{PR}^n$ = $\mathfrak{PR}^n \subset \mathbb{C}^n \mid M \in \mathfrak{PR}^n$, $M + M^* \ge 0$ ($\mathfrak{PR}^1$ is the "right half plane")
- $\mathbb{PR}^n$ = $\mathbb{PR}^n \subset \mathbb{C}^n\mathbb{R}_+ \mid M + M^* \ge 0$ ($\mathbb{PR}^1$ is the "right half space")

In the scalar case, the superscript $n = 1$ will generally be dropped.

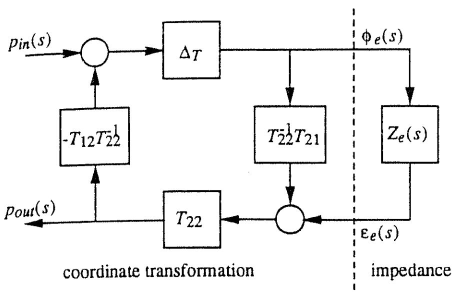

Fig. 2

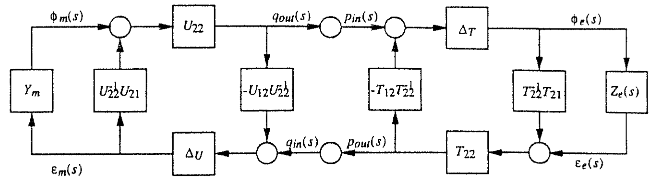

Fig. 3

Impedances and admittances arise when efforts and flows are used to describe bilateral interaction—yet, there are many other legitimate coordinate selections.

III Coordinate Transformations

Consider the change of coordinates from $[\phi_e\ \epsilon_e]^T$ to $[p_{\text{in}}\ p_{\text{out}}]^T$ defined by the nonsingular linear transformations:

$$\begin{bmatrix} p_{\text{in}} \\ p_{\text{out}} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \phi_e \\ \epsilon_e \end{bmatrix} \tag{3a}$$

and

$$\begin{bmatrix} \phi_e \\ \epsilon_e \end{bmatrix} = \begin{bmatrix} \Delta_T & -\Delta_T T_{12} T_{22}^{-1} \\ -T_{22}^{-1} T_{21} \Delta_T & T_{22}^{-1} + T_{22}^{-1} T_{21} \Delta_T T_{12} T_{22}^{-1} \end{bmatrix} \begin{bmatrix} p_{\text{in}} \\ p_{\text{out}} \end{bmatrix} \tag{3b}$$

Where $\Delta_T = (T_{11} - T_{12} T_{22}^{-1} T_{21})^{-1}$. If $\phi_e(s)$ and $\epsilon_e(s)$ are related by an impedance operator $Z_e(s)$, then $p_{\text{in}}(s)$ and $p_{\text{out}}(s)$ are related as follows (see Fig. 2 for the associated block diagram):

$$\begin{aligned} p_{\text{out}}(s) &= (T_{21} + T_{22} Z_e(s))(T_{11} + T_{12} Z_e(s))^{-1} p_{\text{in}}(s) \\ &= T(Z_e(s)) p_{\text{in}}(s) \\ &= A_e(s) p_{\text{in}}(s) \end{aligned} \tag{4}$$

The operator $A_e(s)$ is a linear fractional transformation (LFT) of $Z_e(s)$. An LFT is a conformal map which takes circular regions of the Z_e-plane to circular regions of the A_e-plane, where a half-plane is considered a circle of infinite extent.

Example 2. A well-known and much-used coordinate transformation is from root power variables (efforts and flows) to scattering variables (Anderson and Vongpanitlerd, 1973; Fettweis and Meerkotter, 1977; Paynter and Busch-Vishniac, 1988):

$$\begin{bmatrix} p_{\text{in}} \\ p_{\text{out}} \end{bmatrix} = \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \begin{bmatrix} \phi_e \\ \epsilon_e \end{bmatrix} \tag{5}$$

The associated LFT (known as the Mobius or bilinear transformation) transforms the impedance operator to a scattering operator, $S_e(s)$:

$$S_e(s) = (Z_e(s) - I)(Z_e(s) + I)^{-1} \tag{6}$$

This LFT maps the right half Z_e plane ($\mathcal{PR}^n$) to the interior of the unit disk ($\mathcal{D}_1^n$) in the S_e plane. Thus, in terms of scattering operators, a passive system is bounded in magnitude and completely uncertain in phase. ■

The coordinates used to describe the manipulator must also be changed if the stability properties of the original feedback loop (Fig. 1) are to be retained. Suppose the coordinate transformation applied to the manipulator takes the form:

$$\begin{bmatrix} q_{\text{in}} \\ q_{\text{out}} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \epsilon_m \\ \phi_m \end{bmatrix} \tag{7a}$$

and

$$\begin{bmatrix} \epsilon_m \\ \phi_m \end{bmatrix} = \begin{bmatrix} \Delta_U & -\Delta_U U_{12} U_{22}^{-1} \\ -U_{22}^{-1} U_{21} \Delta_U & U_{22}^{-1} + U_{22}^{-1} U_{21} \Delta_U U_{12} U_{22}^{-1} \end{bmatrix} \begin{bmatrix} q_{\text{in}} \\ q_{\text{out}} \end{bmatrix} \tag{7b}$$

where $\Delta_U = (U_{11} - U_{12} U_{22}^{-1} U_{21})^{-1}$. Figure 3 represents the transformed, coupled system, which must exhibit the same closed-loop characteristic equation as Fig. 1. In effect, this requires that the "junction relation" between $[\phi_m\ \epsilon_m]^T$ and $[\phi_e\ \epsilon_e]^T$ be the same in both cases:

$$\begin{bmatrix} \epsilon_m \\ \phi_m \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} \phi_e \\ \epsilon_e \end{bmatrix} \qquad \text{(untransformed)} \tag{8a}$$

$$U \begin{bmatrix} \epsilon_m \\ \phi_m \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} T \begin{bmatrix} \phi_e \\ \epsilon_e \end{bmatrix} \qquad \text{(transformed)} \tag{8b}$$

Thus, the relationship between U and T is:

$$U \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} T \tag{9}$$

which may be rewritten:

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} T_{22} & -T_{21} \\ -T_{12} & T_{11} \end{bmatrix} \tag{10}$$

Example 2 (Continued). In the case of scattering variables, the manipulator coordinates should be transformed as:

$$\begin{bmatrix} q_{\text{in}} \\ q_{\text{out}} \end{bmatrix} = \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \begin{bmatrix} \epsilon_m \\ \phi_m \end{bmatrix} \tag{11}$$

leading to:

$$S_m(s) = (Y_m(s) - I)(Y_m(s) + I)^{-1} \tag{12}$$

Now, if the environment is passive ($\|S_e\|_\infty \le 1$), a sufficient condition for stability is: $\|S_m\|_\infty < 1$ (i.e., that the manipulator be strictly passive). It is a well-known result that the interconnection of a passive and a strictly passive system is stable (Colgate, 1992). Strict passivity is also a necessary condition for coupled stability if the environment may be any passive system. ■

The general application of coordinate transformations is summarized in the following theorem:

Theorem 1. Consider a class of environments $\mathbb{S}_e^n$ and a nonsingular coordinate transformation T such that $A_e = T(Z_e \in \mathbb{S}_e^n) \in \mathbb{D}_1^n$. T must be a $2n \times 2n$ linear, time-invariant

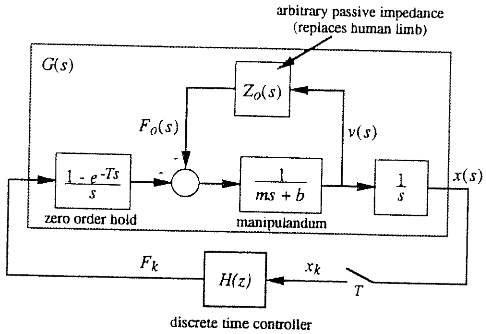

Fig. 4

operator. Suppose that there exists some $Z_1 \in \mathbb{S}_e^n$ such that $A_1 = T(Z_1)$ is stable. Then sufficient conditions for coupled stability are that the manipulator admittance, Y_m, satisfy: $U(Y_m) = A_m \in \mathbb{D}_1^n$; A_m stable, where U is specified in Eq. (10).

The proof follows easily from the discussion above and the small gain theorem. One subtley is that $T(Z_e)$ must be stable in order to apply the small gain theorem. It is clearly impractical to verify that $T(Z_e)$ is stable for every member of $\mathbb{S}_e^n$, but it follows from the "boundary crossing theorem" (Barmish, 1989) that this condition holds if at least one member of $\mathbb{S}_e^n$ is stable when transformed and if $\|T(Z_e)\|_\infty$ is finite for all $Z_e \in \mathbb{S}_e^n$. The former condition is included in the statement of the theorem, while the latter condition follows from the fact that $\mathbb{S}_e^n$ is mapped to the unit cylinder.

The conditions of the theorem are *necessary* as well if the transformed class of environments ($T(\mathbb{S}_e^n)$) completely covers the boundary of the unit cylinder ($\mathbb{D}_1^n$). Due to the nature of the LFT, this is possible only if $\mathbb{S}_e^n$ is a circle or half plane at each frequency. In general, conservativeness is minimized when T maps $\mathbb{S}_e^n$ to the largest possible region bounded by the unit cylinder.

Example 3. Figure 4 illustrates a coupled stability problem which arises in human-machine interaction. The plant is a one degree-of-freedom "manipulandum," which is coupled to a human limb and equipped with a discrete time controller (see, for example, Millman and Colgate, 1991). The human limb is highly uncertain, and is replaced here with an arbitrary passive impedance, $Z_0 \in \mathbb{PR}$ (rationale is provided in Colgate et al., 1993)). The task is to place constraints on the controller $H(z)$ that will guarantee coupled stability.

It can be shown that the closed loop characteristic equation is:

$$1 + H(e^{sT})(1 - e^{-sT})\frac{1}{T}\sum_{n=-\infty}^{n=\infty}\frac{G_1(s + j\omega_s n)}{(s + j\omega_s n)^2} = 0 \qquad (13)$$

where

$$G_1(s) = \frac{1}{ms + b + Z_0(s)} \qquad (14)$$

and $\omega_s = 2\pi/T$ is the sampling frequency. Note that G_1 is an LFT of Z_0. Because $Z_0(j\omega)$ lies in the right half plane for all ω, $G_1(j\omega)$ will lie within a disk for all ω. It is easily shown that this disk, $\mathbb{S}_{G_1}$, has a center at $1/2b$ on the real axis, and a radius of $1/2b$. Because this disk is independent

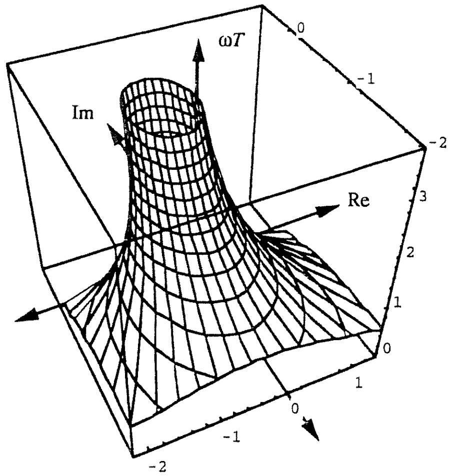

Fig. 5(*a*) Plot of $\mathbb{S}_{G_2}$ for example 3

of frequency, it is possible to extract it from the infinite summation and define a frequency-dependent region, $\mathbb{S}_{G_2}$:

$$\mathbb{S}_{G_2}(\omega) = r(j\omega)\mathbb{S}_{G_1} \qquad (15)$$

where

$$r(j\omega) = \frac{1 - e^{-j\omega T}}{T}\sum_{n=-\infty}^{n=\infty}\frac{1}{(j\omega + j\omega_s n)^2} \qquad (16a)$$

$$r(j\omega) = (e^{-j\omega T} - 1)\frac{T}{4}csc^2\left(\frac{\omega T}{2}\right) \qquad (16b)$$

The multiplier $r(j\omega)$ may be viewed as a frequency-dependent scaling and rotation (due to sampling, it is periodic with period T); therefore, $\mathbb{S}_{G_2}$is a disk at each frequency. Figure 5(*a*) is a plot of the associated region $\mathbb{S}_{G_2}$ in $\mathbb{C}^1\mathbb{R}_+$ space. $\mathbb{S}_{G_2}$ may be mapped to $\mathbb{D}_1$ by the following coordinate change:

$$T = \begin{bmatrix} r(s) & 0 \\ -r(s) & 2b \end{bmatrix} \qquad (17)$$

It is easily verified that $T(G_2(s))$ is stable for all $G_2 \in \mathbb{S}_{G_2}$. Theorem 1, therefore leads to the following coupled stability criterion:

$$\|U(H(e^{sT})\|_\infty = \left\|\frac{r(s)H(e^{sT})}{r(s)H(e^{sT}) + 2b}\right\|_\infty < 1 \qquad (18)$$

It can be shown that this condition implies the stability of $U(H(e^{sT}))$, which is also required to satisfy Theorem 1.

By mapping the boundary of $\mathbb{D}_1$ through U^{-1}, the region of $\mathbb{C}^1\mathbb{R}_+$ space where $H(e^{j\omega T})$ may lie can be explicitly displayed, as in Fig. 5(*b*). An application of this criterion is reported in (Colgate et al., 1993). ■

This example illustrates that the feedback loop does not need to be cut between the manipulator and environment, as it was in Fig. 1. As in this example, it is often convenient to lump manipulator dynamics with the environment while isolating the feedback controller.

IV Noncircular Environment Classes

In practice, disk-shaped environments usually stem from knowledge of passivity, or from high frequency uncertainty

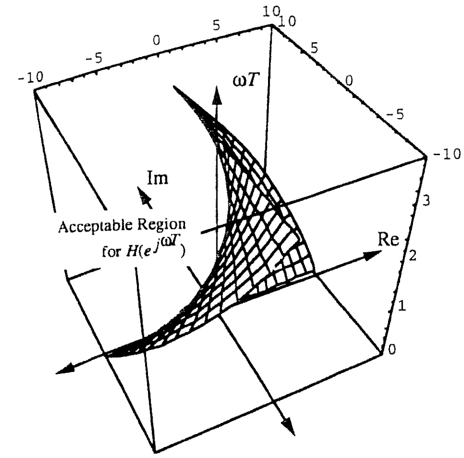

Fig. 5(b) The Nyquist plot of a controller $H(z)$ that satisfies Eq. (18) will lie in the indicated half-space

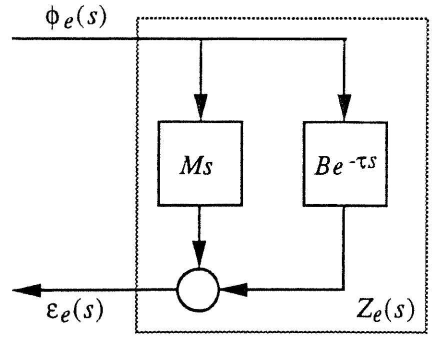

Fig. 6

(unmodeled dynamics). Noncircular environment classes also arise in a number of instances, for example when the class is defined by parameter intervals. In such cases, Theorem 1 will produce conservative results, because the noncircular class must first be captured within a disk. Theorem 2 provides a means of reducing this conservativeness.

Theorem 2. Consider the feedback system in Fig. 1. Suppose that $Z_e \in \mathbb{S}_e^n$, that $\mathbb{S}_e^n$ is a simply connected set, and that the feedback system is known to be stable for some $Z_1 \in \mathbb{S}_e^n$. Suppose also that at each frequency ω:

$$\mathbb{S}_e^n \subset [[\cdots[[\mathbb{S}_1^n \circ \mathbb{S}_2^n] \circ \mathbb{S}_3^n]\cdots] \circ \mathbb{S}_p^n] \tag{19}$$

where $\circ$ represents either $\cap$ or $\cup$, and where associated with each $\mathbb{S}_i^n$ is an LFT T_i such that $\mathcal{D}_1^n = T_i(\mathbb{S}_i^n)$. Define U_i in the usual way. Then a sufficient condition for coupled stability is:

$$m[\cdots m[m[\bar{\sigma}(U_1(Y_m(j\omega))), \bar{\sigma}(U_2(Y_m(j\omega)))], \bar{\sigma}(U_3(Y_m(j\omega)))]\cdots, \bar{\sigma}(U_p(Y_m(j\omega)))] < 1 \ \forall\ \omega \tag{20}$$

where m represents the minimum of its arguments if the associated operator in Eq. (19) is $\cap$, and the maximum if the operator is $\cup$.

In the SISO case, an equivalent condition is that $Y_m(j\omega)$ lie strictly within the following region at each frequency:

$$\mathbb{S}_m^1 = [[\cdots[[U_1^{-1}(\mathcal{D}_1^1)\cdot U_2^{-1}(\mathcal{D}_1^1)]\cdot U_3^{-1}(\mathcal{D}_1^1)]\cdots]\cdot U_p^{-1}(\mathcal{D}_1^1)] \tag{21}$$

where $\cdot = \cup$ if $\circ = \cap$, and vice versa.

Proof. Because stability is achieved with Z_1, and the set of environments is simply connected, it is necessary to show only that $\det(I + Z_e(j\omega)Y_m(j\omega)) \neq 0$ for any ω and for any $Z_e(s) \in \mathbb{S}_e^n$ (Maciejowski, 1989). The idea is that there cannot exist $Z_1(j\omega)Y_m(j\omega)$ and $Z_2(j\omega)Y_m(j\omega)$ which exhibit different numbers of encirclements of -1 without there being $Z_3(j\omega)Y_m(j\omega)$ which passes through -1.

Suppose that $Z_e(j\omega_1) \in \mathbb{S}_1^n \cap \mathbb{S}_2^n$, and that associated with both regions are LFTs (T_1 and T_2, respectively), that map them to $\mathcal{D}_1^n$. Suppose further that $\bar{\sigma}(U_1(Y_m(j\omega_1))) < 1$. This ensures that $\det(I + Z_e(j\omega_1)Y_m(j\omega_1)) \neq 0$. But so does the condition $\bar{\sigma}(U_2(Y_m(j\omega_1))) < 1$, so that it suffices to test only the minimum of the two singular values. If $Z_e \in \mathbb{S}_1^n \cup \mathbb{S}_2^n$, $\bar{\sigma}(U_1(Y_m(j\omega_1))) < 1$ ensures that no member of $\mathbb{S}_1^n$ can make the determinant zero, while $\bar{\sigma}(U_2(Y_m(j\omega_1))) < 1$ does the same for $\mathbb{S}_2^n$. Use of the maximum is a means of requiring that both conditions are met. This reasoning is readily propagated to more general logical combinations of regions.

Proof of the geometric condition for SISO systems follows easily, and will not be given here. ■

Equation (20) is a necessary condition for coupled stability as well, if the class of environments completely covers the boundary of $\mathbb{S}_e^n$.

Example 4 is somewhat contrived, but has two features to recommend its use here: it is simple (the necessary sets are readily constructed and viewed in the complex plane), and it illustrates several different types of logical combinations.

Example 4. Consider the coupled system shown in Fig. 1, in which the manipulator and environment are 1-ports. Suppose that the environment is a velocity-controlled inertia (M), as shown in Fig. 6, with uncertain feeback gain (B) and delay (τ). Further, suppose that the following limits can be placed on B and τ:

$$0 \leq B \leq B_{\max} \tag{22a}$$

$$0 \leq \tau \leq \tau_{\max} \tag{22b}$$

and that $B_{\max}/M < \pi/\tau_{\max}$. Then at any frequency $\omega < 2\pi/\tau_{\max}$, the class of environments ($\mathbb{S}_e$) may be described as a pie-shaped region that can be formed by the logical combination of two half-planes ($\mathbb{S}_{\rm I}$ and $\mathbb{S}_{\rm II}$) and a disk ($\mathbb{S}_{\rm III}$) (see Fig. 7). Associated with each of these regions is an LFT ($T_{\rm I} - T_{\rm III}$) that maps it to $\mathcal{D}_1$, as well as an inverse LFT $U_{\rm I}^{-1} - U_{\rm III}^{-1}$) that maps $\mathcal{D}_1$ to a region ($\mathcal{R}_{\rm I} - \mathcal{R}_{\rm III}$) in the manipulator admittance plane. The region of acceptable manipulator admittances ($\mathcal{R}_m$) is then formed by the appropriate logical combination. For instance, if $\omega < \pi/T_{\max}$, the combinations are:

$$\mathbb{S}_e = \mathbb{S}_{\rm I} \cap \mathbb{S}_{\rm II} \cap \mathbb{S}_{\rm III} \tag{23a}$$

$$\mathcal{R}_m = \mathcal{R}_{\rm I} \cup \mathcal{R}_{\rm II} \cup \mathcal{R}_{\rm III} \tag{23b}$$

Combinations which arise at higher frequencies are illustrated in Fig. 7. Because the region $\underline{\mathcal{R}}_m$ is open at each frequency, the complementary region, $\mathcal{R}_m$, is shown.

For the purposes of computation, the coupled stability criterion may be written as:

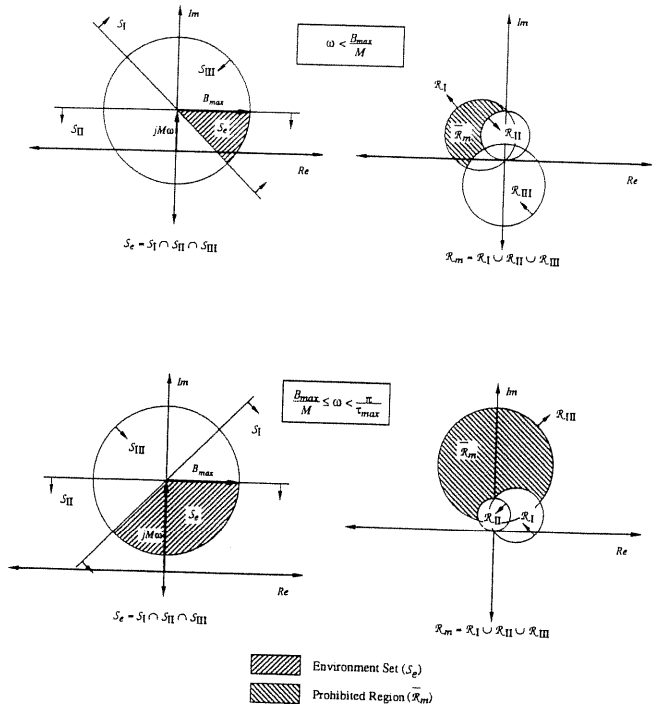

Fig. 7 Graphical construction of environment sets (S_e) and manipulator sets ($\mathcal{R}_m$) for Example 4. Arrows point to region interiors, which in some cases are circle exteriors.

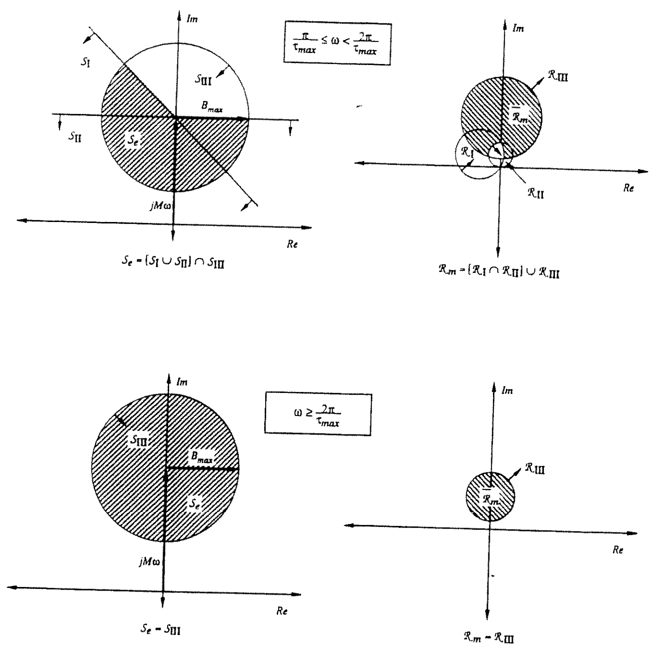

Fig. 7 (Continued)

$$\min[|U_{\rm I}(Y_m(j\omega))|, |U_{\rm II}(Y_m(j\omega))|, |U_{\rm III}(Y_m(j\omega))|] < 1 \quad 0 < \omega < \pi/T_{\max}$$

$$\min[\max[|U_{\rm I}(Y_m(j\omega))|, |U_{\rm II}(Y_m(j\omega))|, |U_{\rm III}(Y_m(j\omega))|] < 1 \quad \pi/T_{\max} \le \omega < 2\pi/T_{\max}$$

$$|U_{\rm III}(Y_m(j\omega))| < 1 \quad 2\pi/T_{\max} \le \omega < \infty$$

It is also necessary to ensure that the coupled system is stable for some member of the class of environments, for example $Z_1(s) = Ms + B_{\max}$. ■

V Discussion

Two powerful techniques for reducing conservativeness in coupled stability criteria were introduced: coordinate transformations and logical operations.

The first use of coordinate transforms to strengthen stability criteria was probably Zames' work (Zames, 1966a, Zames, 1966b), preliminary to his famous "circle criterion." Zames introduced a very simple LFT that enabled him to bound the extended L_2 norm of certain nonlinearities with that of a disk-shaped set of linear operators. Zames introduced the term "conic sector" to describe the sets, and this term remains prevalent in the literature. Safonov and Athans extended Zames work to certain multivariable feedback systems, and to uncertain LTI feedback systems (Safonov and Athans, 1981). In the latter work, the authors retain the term conic sector, but permit the associated cone (disk) center and radius to be frequency-dependent. It is interesting to note that the cone center is generally associated with the notion of a nominal model, while the radius is a measure of uncertainty. The approach of coordinate transformations, presented here, leads to similar results, but owes its genesis to the idea of scattering variables (the class of transformations presented here can be viewed as a generalization of the Mobius transformation). One benefit of the coordinate transform approach is that half planes are naturally incorporated. This is not so in the conic sectors approach, which works with disks only.[3] This limitation has been addressed by Safonov (Safonov et al., 1987).

The use of logical combinations has no clear precedent in the literature. This may be because conic sector theory was developed largely in support of the absolute stability problem, for which only disks are relevant. Theorem 2 appears to be primarily applicable to coupled stability problems in which constraints other than passivity can be placed on the class of environments. Application to stability problems which arise in human-machine interaction is now underway.

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[3] A disk of very large radius approximates a half plane, but the associated transform is poorly conditioned (Safonov et al., 1987).

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