Coupled Stability of Multiport Systems—Theory and Experiments

This paper presents both theoretical and experimental studies of the stability of dynamic interaction between a feedback controlled manipulator and a passive environment. Necessary and sufficient conditions for "coupled stability"—the stability of a linear, time-invariant n-port (e.g., a robot, linearized about an operating point) coupled to a passive, but otherwise arbitrary, environment—are presented. The problem of assessing coupled stability for a physical system (continuous time) with a discrete time controller is then addressed. It is demonstrated that such a system may exhibit the coupled stability property; however, analytical, or even inexpensive numerical conditions are difficult to obtain. Therefore, an approximate condition, based on easily computed multivariable Nyquist plots, is developed. This condition is used to analyze two controllers implemented on a two-link, direct drive robot. An impedance controller demonstrates that a feedback controlled manipulator may satisfy the coupled stability property. A LQG/LTR controller illustrates specific consequences of failure to meet the coupled stability criterion; it also illustrates how coupled instability may arise in the absence of force feedback. Two experimental procedures—measurement of endpoint admittance and interaction with springs and masses—are introduced and used to evaluate the above controllers. Theoretical and experimental results are compared.

1 Introduction

In this paper, the stability of dynamic interaction between a feedback-controlled physical system and a passive environment is investigated. New theoretical and experimental results are presented. This work is motivated largely by the problem of controlling the interaction between a robot and a dynamic environment; i.e., "force control," "compliance control," or "impedance control" (An, 1986; Hogan, 1985; Mason, 1981; Whitney, 1987). One of the more challenging problems to arise in this research is that of coping with "contact instability." Contact instability is, as its name implies, the phenomenon of loss of stability that occurs when a robot contacts, or couples to, a dynamic environment (Colgate, 1989; Eppinger and Seering, 1992, Kazerooni et al., 1990; Wlassich, 1986). It is frequently observed when a force feedback controlled robot contacts a rigid surface. Indeed, contact instability has often been assumed to be a consequence of force feedback; however, there are many other instances in which coupling to a dynamic environment may compromise stability, as will be demonstrated in this paper. A review of the literature pertaining to contact stability is given in (Hogan and Colgate, 1989).

The contributions of this paper are:

- Previous results providing necessary and sufficient conditions for 1-port coupled stability—the stability of a controlled system coupled at a single port to an arbitrary passive environment—are extended to n-port coupled stability.
- It is demonstrated that discrete-time controller implementation does not necessary prohibit coupled stability from being achieved.
- An approximate condition for the n-port coupled stability of a physical system (continuous time) with a discrete time controller is presented.
- The application and utility of these results are demonstrated via the analysis of two robot controllers.
- Experimental confirmation and comparison to the analytical results are provided.

2 Background: Results With 1-Ports

This section offers a brief review of several theoretical results concerning the stability of a linear feedback con-
trolled system interacting with (i.e., mechanically coupled to) a passive environment at a single port. All proofs and derivations have been omitted; these may be found in (Colgate and Hogan, 1987; Colgate and Hogan, 1988). Linear 1-port passivity may be defined as follows (Brune, 1931):

1-Port Passivity. A linear time-invariant 1-port is passive if:

1. \( Z(s) \) has no poles in the open right half plane.
2. \( Z(s) \) has a Nyquist plot which lies wholly within the closed right half plane.

where \( Z(s) \) is the impedance or admittance at the port. A function \( Z(s) \) which satisfies these conditions is termed positive real. This definition and the Nyquist stability criterion may be used to prove the following result (Colgate and Hogan, 1988):

1-Port Coupled Stability Criterion I. A necessary and sufficient condition to ensure the stability of a linear, time-invariant, stable plant coupled at a single interaction port to an environment which is stable and passive, but otherwise arbitrary (linear or nonlinear), is that the impedance (or admittance) of the plant be positive real.

In other words, the plant, although feedback controlled, should have the interaction port behavior of a passive system in order to guarantee coupled stability.

The following alternative coupled stability criterion identifies a reduced class of environments which is sufficient to test for coupled stability (Colgate, 1988; Colgate and Hogan, 1988):

1-Port Coupled Stability II. A necessary and sufficient condition to guarantee the stability of a stable LTI plant coupled at a single interaction port to an arbitrary passive environment, is that the plant be stable when coupled to all ideal linear 1-port inertias and all ideal linear 1-port springs.

Close examination of the coupled stability criteria given above reveals that they guarantee only that the energy of the coupled system is non-increasing, not that this energy decreases with time. Accordingly, these criteria guarantee that, if the environment is linear, the states of the coupled system do not grow exponentially, but do not guarantee that these states decay exponentially. If a guarantee of exponential stability is desired, a somewhat stronger criterion is needed. This can be achieved by requiring that the manipulator’s impedance be “strictly positive real” (i.e., dissipative) according to any of a number of definitions (Wen, 1988). However, before taking such an approach, one should carefully consider whether it is necessarily desirable that all states decay exponentially. For instance, suppose that one state represents vertical deflection of the robot’s endpoint. If the robot’s steady-state behavior is spring-like, then under a constant gravitational load, this deflection will take on a finite value and will not decay to zero. In order to accommodate such cases, and yet ensure that all velocities are exponentially stable, a new definition of “strictly positive real” was recently proposed (Colgate, 1992). Such a definition will not, however, be used in this paper.

3 Multiport Coupled Stability

Preliminaries. The development of a multiport coupled stability criterion depends on the concept of a hybrid matrix, and on the condition for passivity of a multiport. Hybrid matrices arise because a multiport system cannot always be represented as an impedance matrix or an admittance matrix. Some ports may demand impedance causality while others demand admittance. It is always possible, however, to represent a multiport with a hybrid matrix, in which each port takes on its own preferred or required causality (Anderson et al., 1966).

The conditions for multiport passivity are (Anderson and Vongpanitlerd, 1973):

n-Port Passivity. A linear time-invariant n-port is passive if:

1. \( H(s) \) has no poles in the open right half plane.
2. The Nyquist plots of the n upper left determinants of \( H(s) + \hat{H}(s) \) lie wholly in the closed right half plane.

where \( H(s) \) is the hybrid matrix and the tilde indicates the complex conjugate transpose. A matrix which satisfies these conditions is termed “positive real.”

A second version of n-port passivity which is useful in proving the multiport coupled stability criterion is given in the Appendix. The multiport coupled stability theorem may now be stated in terms of hybrid matrices and n-port passivity:

### Nomenclature

\[
\begin{align*}
\hat{A} & = \text{complex conjugate transpose of } A \\
A^* & = \text{sampled signal, pulse transfer function} \\
A & = \text{vector} \\
B & = \text{damping coefficient; velocity feedback gain matrix} \\
b & = \text{damping coefficient} \\
D(s) & = \text{“determinant term” in multiport coupled stability test} \\
e & = \text{error; effort} \\
f & = \text{flow} \\
F & = \text{force vector} \\
G_{xy}(\omega) & = \text{auto } (i = j) \text{ or cross } (i \neq j) \text{ spectral density, position data} \\
G_{xyi}(\omega) & = \text{cross spectral density, position, and force data} \\
H & = \text{hybrid matrix; transfer function} \\
H_{2OH}(s) & = \text{zero-order hold transfer function} \\
I & = \text{inertia matrix} \\
J & = \text{Jacobian} \\
K & = \text{stiffness; displacement feedback gain matrix} \\
m, M & = \text{mass} \\
T & = \text{time delay, sampling period} \\
U & = \text{control vector} \\
I_0 & = \text{ component of admittance matrix} \\
Z & = \text{impedance} \\
\theta & = \text{joint angles} \\
\omega & = \text{frequency; joint angular velocities} \\
\omega_N & = \text{Nyquist frequency} \\
\omega_s & = \text{sampling frequency} \\
\tau & = \text{time constant}
\end{align*}
\]
Theorem (Multiport Coupled Stability). If $H_1$ is a linear time-invariant n-port which is coupled to a passive, but otherwise arbitrary, n-port, then a necessary and sufficient condition to guarantee the stability of the coupled system is that the hybrid matrix $H_1(s)$ be positive real. The proof of this theorem is given in the appendix.

4 Coupled Stability With Discrete-Time Control

The conditions for passivity of a linear time-invariant plant do not strictly apply if the feedback controller for the plant is implemented in discrete time. This is because, even when analog data is properly filtered, the act of sampling creates new frequencies. Using the impulse-train modulation model (Tou, 1959), the spectrum of a continuous time signal, $y(t)$, sampled at a frequency $\omega_s = 2\pi/T$, is:

$$y^*(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} y(j\omega + jk\omega_s)$$

where the asterisk indicates a sampled signal. If $y(t)$ contains no frequencies above the Nyquist frequency, then the spectrum will consist of a fundamental and an infinite series of identical, but distinct, side lobes. The sampled input will be operated on by a discrete time compensator and, typically, a zero-order hold to produce an output command to the plant. The hold, and possibly the compensator, will filter the side lobes; however, the output will contain power at frequencies other than those contained in $y(t)$.

Because new frequencies are created, the continuous time theories of passivity and coupled stability do not apply, even though the plant-environment interaction is fundamentally continuous. Below, two approaches to assessing coupled stability under these conditions are considered.

Direct Analysis of Coupled Stability. For a simple enough plant and discrete-time controller, it is possible to prove that coupled stability may be achieved. Consider, for example, the control system shown in Fig. 1. Here, the plant is a damped inertia and the control is proportional velocity feedback, implemented in discrete time (sample rate $T$).

First, consider the admittance of the plant/environment combination:

$$Y_c(s) = \frac{1}{ms + b + Z_c(s)}$$

It is straightforward to show that, if $Z_c(s)$ is passive, the Nyquist plot of $Y_c(s)$ must lie within a disk of radius $1/2b$, centered at $(1/2b, 0)$. Thus, $|Y_c(j\omega)| \leq 1/b$ for all $\omega$.

Next, consider the open-loop transfer function, $G^*(s)$, from $e(s)$ to $v^*(s)$. The frequency response associated with $G^*(s)$ is (Tou, 1959):

$$G^*(j\omega) = \frac{B}{T} \sum_{k=-\infty}^{\infty} H_{ZOH}(j\omega - jk\omega_s) Y_c(j\omega - jk\omega_s)$$

$$= \frac{B}{T} \sum_{k=-\infty}^{\infty} Y_c(j\omega - jk\omega_s) \left( 1 - e^{-j\omega T} \right)$$

Thus, the magnitude of $G^*(j\omega)$ can be bounded:

$$|G^*(j\omega)| \leq \frac{B}{bT} \sum_{k=-\infty}^{\infty} \frac{1}{1 - e^{-j\omega T}}$$

It can be shown that the summation is finite (except when $\omega = 0$, at which point L'Hopital's rule shows that the expression as a whole is finite), thus ensuring boundedness (Schenkel, 1993). Coupled stability may therefore be guaranteed by choosing $B$ sufficiently small that the Nyquist plot of $G^*(j\omega)$ cannot encircle -1.

This example has demonstrated that it is possible for a physical system with discrete time control to exhibit the coupled stability property. This example has not, however, illustrated a practical means of assessing coupled stability. To compute the limiting value of $B$ will require, in principle, summation of an infinite series of regions in the Nyquist plane. Even the computation of a finite approximation represents a nontrivial exercise. Therefore, a much simpler approximate method is considered below.

Approximate Method. The method introduced here is based on the assumption that the intrinsic dynamics of the plant act as a low pass filter with a cutoff well below the Nyquist frequency. Thus, higher frequencies due to sampling can be ignored. There are surely instances in which this is a poor assumption, but in the work presented here it is found that manipulator inertia dominates the measured admittances at frequencies above 8 Hz, while the Nyquist frequency is 50 Hz.

The problem remains of how best to approximate the impedance or admittance of a continuous time plant with a discrete time controller. Here, the following approach is taken: a closed-loop, continuous time admittance, $Y(s)$, is assumed to exist, embedded between a zero-order hold at the input, and an ideal sampler at the output. Note that these are the input and output at the interaction port, and in reality neither the hold nor the sampler exists. A pulse-transfer function, $Y^*(z)$, is then found analytically by discretizing the plant and introducing the discrete-time compensator. Any of a number of commercial software packages will compute the frequency response of a pulse-transfer function, which consists of the gain and phase of $Y^*(e^{j\omega})$ for $\omega$ in the range $0 < \omega < \omega_N$, where $\omega_N$ is the Nyquist frequency. As a final step, the gain and phase due to the zero-order hold and the sampler are subtracted to yield the approximate frequency response at frequencies up to $\omega_N$.

At sufficiently low frequencies, the combination of zero-order hold and sampler behaves simply as a delay of $T/2$, so that the magnitude scaling is unity, and the phase lag is $\omega T/2$ radians. For the cases considered in this paper, this is an excellent approximation, so that it is a simple matter to correct the result of the pulse-transfer function analysis by subtracting the phase lag $\omega T/2$. 
5 Coupled Stability Analysis of Controllers for a Two-Link Manipulator

As a means of illustrating the application and utility of the coupled stability criteria presented above, two controllers for a two-link, revolute joint, direct drive manipulator have been designed, analyzed, and implemented. In this section, the controller designs are presented and their coupled stability properties analyzed. In Section 6, the experimental evaluation of these same controllers is discussed.

5.1 Impedance Control. In (Hogan, 1985), Hogan describes an impedance control implementation that uses a form of feedback linearization to provide a linear, time-invariant impedance for a nonlinear robot. Such an implementation would, of course, make it possible to apply the techniques presented here directly to the passivity analysis of the robot. Moreover, the analysis would be trivial, because Hogan’s method, which assumes rigid links and no actuator dynamics, enables a complete specification of endpoint impedance; thus, the impedance can always be chosen to be passive. This implementation, however, is computationally intensive, and for the purposes of this work, a simpler approach is taken: the dynamics of the two-link manipulator are linearized about zero velocity operating points at various locations within the workspace. In addition to simplifying the real-time implementation, this puts the impedance controller on an equal footing with the LQG/LTR controller described below.

The linearized dynamics may be written:

$$\frac{d}{dt} \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\hat I^{-1} \beta \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \hat I^{-1} \end{bmatrix} U + \begin{bmatrix} 0 \\ \hat I^{-1} J^T \end{bmatrix} F$$

where \(I\) is the inertia matrix at the operating point (nominal configuration), \(J\) is the Jacobian at the operating point, \(\beta\) is a matrix of joint damping coefficients, and where \(\theta\) and \(\omega\) are understood to be vectors of small displacements from the nominal joint angle \(\theta_0\) and joint angular velocity \(\omega_0 = \theta\) vectors. A more compact notation is:

$$\Theta = \mathbf{A} \Theta + BU + LF$$

In order to design an impedance controller for this linearized robot, the desired endpoint behavior (desired impedance) is first specified as:

$$F = -K_\xi x - B_\xi v$$

where \(K_\xi\) and \(B_\xi\) are positive definite stiffness and damping matrices, and \(x\) and \(v\) are vectors of small displacements from the nominal endpoint location and velocity (which is zero). The principle of virtual work may be used to relate quasistatic joint torques to endpoint forces: \(U = J^T F\). Using this relation, the Jacobian relation, and adding a term to compensate for viscous losses in the manipulator, the impedance control law is found to be:

$$U = -\hat J^T K_\xi \dot \theta + (J^T B_\xi J + \beta) \omega$$

Figure 2 is a multivariable Nyquist plot of the admittance of the closed-loop, impedance controlled manipulator (discrete time control implementation; sample rate of 100 Hz), obtained with the approximate method described in Section 4. The behavior is clearly passive. This result is by no means surprising. Due to the colocation of the actuators and sensors at the robot’s joints, position and velocity feedback essentially mimic multivariable springs and dampers, respectively. Variations on this basic approach, such as PD control, will exhibit equivalent coupled stability properties.

It remains questionable, however, whether this prediction of passivity is a consequence of a too-simple model. Real effects, including discrete-time control implementation, unmodeled dynamics, and actuator/sensor noncollocation, can potentially lead to non-passive behavior. Can an impedance controlled manipulator, such as that described here, behave passively in actual implementation?

Somewhat surprisingly, the answer appears to be yes. This is because of the role that intrinsic manipulator dynamics play in mitigating the above effects. For instance, it was shown in Section 4 that manipulator damping plays a key role in preserving coupled stability when feedback control is implemented in discrete time. Intrinsic manipulator dynamics—which are passive—also tend to dominate the closed loop admittance at higher frequencies where unmodeled dynamics are important. This has been discussed in detail in (Colgate, 1989). Finally, the experimental results presented in Section 6 support the conclusion that, even in the face of real effects, passive behavior can be achieved.

5.2 LTR/LQG Control. This section describes a Linear Quadratic Gaussian, Loop Transfer Recovery (LQG/LTR) controller that has been implemented on the two-link manipulator. Description of the LQG/LTR methodology itself can be found in (Maciejowski, 1989). For the purposes of this paper, a full understanding of the methodology is not necessary; it suffices to understand the following points:

- This implementation of LQG/LTR includes integral action in each control channel.
- The primary goal of the design procedure is to “shape” the frequency domain behavior of the singular values of the loop transfer function. In particular, this imple-
Fig. 3 LQG/LTR control: singular values of the closed-loop transfer function

Fig. 4 LQG/LTR control: Nyquist plots of \( Y_1(\omega) \), \( Y_2(\omega) \), and \( \Delta(\omega) \)

mentation attempts to match the magnitudes of all the singular values, and to make these have a constant slope of \(-20 \text{ dB/decade}\) in the frequency domain, at least for some desired bandwidth. This is achieved by generating an approximate plant inverse.

- To emphasize the multivariable nature of the design, the nominal manipulator configuration is chosen not to be decoupled. The link angles are: \( \theta_1 = 30 \text{ deg}, \theta_2 = 150 \text{ deg} \).
- The formal LQG/LTR design procedure is ill-conditioned for a double-integrator plant (which the manipulator very nearly is). Therefore, the plant is preconditioned by using position and velocity feedback to move the poles into the left half plane.

The LQG/LTR compensator for this problem is eighth order. The closed loop singular values, assuming discrete time implementation and no errors in the plant model, are shown in Fig. 3. The shape of this plot indicates excellent command following with a bandwidth of about 5 Hz. A multivariable Nyquist plot of the corresponding closed loop admittance is shown in Fig. 4. It is evident that the coupled stability property is not achieved. The large left half plane loop in the Nyquist plot occurs at low frequency and is due to integral action; it suggests stability problems upon coupling to masses. The smaller loop occurs for a range of higher frequencies (5.6–9.5 Hz), and suggests stability problems upon coupling to springs with stiffnesses within a certain interval.

The LQG/LTR methodology is intended to provide good servo behavior, not necessarily good interactive behavior. The point of introducing it is to show that the two goals are quite distinct. In addition, this implementation is an example of a robot controller without force feedback that can exhibit instability upon contact with a stiff surface.

6 Experiments With a Two-Link Manipulator

Two types of experiments are reported in this section: measurement of the two-link manipulator’s endpoint admittance, and interaction with springs and masses. Each of these has been performed with both controllers described in the previous section. The principal objective of these experiments is to provide experimental verification of the relationship between impedance and coupled stability. However, the experiments also lend insight into a number of other important issues. For instance, practical issues affecting the measurement of mechanical admittance have been addressed. Also, the interaction tests provided an opportunity to examine behavior not covered by linear analysis, such as the relationship of contact stability to coupled stability.

6.1 Methods

Manipulator. A complete description of the two-link manipulator used in this study can be found in (Faye, 1986). For the present purposes, it suffices to know that it is a serial, planar manipulator; the proximal link is directly driven, and the distal link is driven by a base-mounted motor via a parallelogram linkage. Each motor is equipped with an optical shaft encoder and a tachometer. A six-axis force sensor is mounted to the manipulator’s endpoint. The manipulator controllers are implemented on a PDP 11/73 digital computer, which is also responsible for data collection. The update rate for all controllers is 100 Hz, while data collection is triggered by interrupts from the force transducer, received at a rate of 104 Hz.

Experimental Procedures (Admittance Measurement). To measure admittance, it is necessary to disturb a system at the interaction port, and to measure both input and output. It is not necessary that the disturbance be created by a pure source, or even a dynamic system with a very high or a very low impedance. For these experiments, the disturbances are provided by a velocity-controlled d.c. motor mounted in a concrete footing. This device is used to spin the manipulator endpoint at a constant velocity (thus, the disturbance generator is called the “spinner”), while endpoint force and joint angle measurements are logged. For each controller, data are taken over a range of spinner velocities (frequencies), for both clockwise and counterclockwise spinning.

A variety of factors affect the usable frequency range. At low enough frequency, the integral action of the LQG/LTR controller is enough to overcome the spinner, while at high enough frequency the force transducer saturates. Both of these factors are functions of the spinning radius, which is adjustable; however, the resolution of the optical encoders (for measuring joint angles) establishes a lower limit on the radius of approximately 6 millimeters. Accounting for these
factors, the useable bandwidth is found to be 0.2—8.0 Hz. This measurement bandwidth is, in fact, the primary consideration in selecting the closed loop bandwidths of the controllers.

Data are collected at some multiple ($n_u \geq 2$) of 9.6 ms, which is the update rate of the force transducer. The multiple $n_u$ is adjusted according to the spinner frequency in order to keep the number of samples per cycle at about 16. Each individual constant velocity record consists of 1024 samples.

One of the challenges in accurate measurement of admittance is accounting for all measurement delays. In these experiments, the largest delay, approximately 9.6 msec, is due to force transducer computation. To compensate for this, the joint angle measurements, for which there is essentially no delay, are placed into temporary storage for one data cycle, creating an artificial delay of 9.6 ms. Additional delay in the force measurement is due to the anti-alias filter. An approximate transfer function for this filter was obtained from the manufacturer, and it was found that its phase lag varies nearly linearly with frequency within the measurement range. Therefore, the filter is treated as a pure delay (of 9 ms), and is compensated for in data processing.

*Interaction Tests.* The interaction tests are of a more qualitative nature than the admittance measurements, primarily due to the difficulty of precisely characterizing the various environments (springs and masses). However, the principal role of these experiments is qualitative: to demonstrate the consequences of meeting or failing to meet the coupled stability criterion. In addition, these experiments provide a crude test of passivity over a frequency range much greater than that obtained from the spinner. Also, interaction tests can be used to explore the relationship of contact stability to coupled stability.

Figure 5 shows the manipulator interacting with a mass. So that the manipulator need not be burdened with weight, the mass is hung from the ceiling. For the small motions in these experiments, the gravitational restoring force is not noticeable. In fact, due to friction at the joints of the manipulator, there is no single equilibrium point. The mass is also free to spin relative to the end of the manipulator. An experiment proceeds by commanding the manipulator to execute a 4 cm diameter, 2 s/cycle circular trajectory, and storing position data for ten seconds. This procedure is repeated for both controllers and varying levels of added mass.

Figure 6 shows the manipulator interacting with a spring. The spring is a simply supported beam; its stiffness may be adjusted by varying the cantilevered length. To investigate coupled stability, the manipulator holds the beam between a pair of pinch rollers. The manipulator is given a ramp input which reaches a plateau at 0.5 cm. The slope of the ramp is variable, but is held at 10 cm/s for most experiments. To investigate contact stability, only a single roller is mounted at the end of the manipulator. It is then commanded to execute the same circular trajectory as in the mass interaction experiments; however, the trajectory intersects the spring.

*Data Analysis (Admittance Measurement).* Data analysis proceeds as follows. The raw joint angle data are first transformed to endpoint positions via the manipulator kinematics. The endpoint forces are transformed from the sensor frame to the lab frame. The mean and linear trends are then removed from all data records. The data are then transformed to the frequency domain via a fast fourier transform (FFT). The FFT algorithm divides each record into four contiguous sections of 256 samples each, as well as an overlapping group of three contiguous sections shifted a half section length relative to the others. Each section is shaped with a Hanning window.

From the frequency domain data, the following auto- and cross-spectral densities are found: $G_{11}(\omega)$, $G_{12}(\omega)$, $G_{11F}(\omega)$, and $G_{12F}(\omega)$. Each of these is sharply peaked at the excitation frequency, where coherences are typically found to be greater than 0.9995. High coherences justify the calculation of linear transfer functions, which are computed point by point in frequency. From a given set of data, the
following transfer functions are computed at a given frequency:

\[ S(\omega) = \frac{G_{x1x2}(\omega)}{G_{x1x1}(\omega)} ; H_1(\omega) = \frac{G_{x1F1}(\omega)}{G_{x1x1}(\omega)} ; H_2(\omega) = \frac{G_{x1F2}(\omega)}{G_{x1x1}(\omega)} \]

Presuming that these transfer functions are based on clockwise rotation, a complementary set, \( S'(\omega), H'_1(\omega), H'_2(\omega) \), can be computed from counterclockwise data. Although the ultimate goal of this procedure is to compute the admittance matrix, transfer functions with position input have been computed since the position records are the most sharply peaked and coherent. Thus, dynamic stiffness must be computed as an intermediate step. The dynamic stiffness matrix is defined as:

\[
\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} K_{11}(\omega) & K_{12}(\omega) \\ K_{21}(\omega) & K_{22}(\omega) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

Its components may be computed as follows:

\[
\begin{bmatrix} \text{Re}K_{11} \\ \text{Im}K_{11} \\ \text{Re}K_{12} \\ \text{Im}K_{12} \\ \text{Re}K_{21} \\ \text{Im}K_{21} \\ \text{Re}K_{22} \\ \text{Im}K_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \text{Re}S & -\text{Im}S \\ 0 & 1 & \text{Im}S & \text{Re}S \\ 1 & 0 & \text{Re}S' & -\text{Im}S' \\ 0 & 1 & \text{Im}S' & \text{Re}S' \\ 1 & 0 & \text{Re}S & -\text{Im}S \\ 0 & 1 & \text{Im}S & \text{Re}S \\ 1 & 0 & \text{Re}S' & -\text{Im}S' \\ 0 & 1 & \text{Im}S' & \text{Re}S' \end{bmatrix}^{-1} \begin{bmatrix} \text{Re}H_1 \\ \text{Im}H_1 \\ \text{Re}H_1' \\ \text{Im}H_1' \\ \text{Re}H_2 \\ \text{Im}H_2 \\ \text{Re}H_2' \\ \text{Im}H_2' \end{bmatrix}
\]

As a final step, the admittance, \( Y(j\omega) \), is found by inverting the dynamic stiffness matrix and multiplying by \( j\omega \) (i.e., differentiating).

Random errors in the admittance matrix solution are found to be small (typically less than ±3 deg in phase), while systematic errors are considerably more troublesome. The greatest source of systematic error is the time delay between force measurement and position measurement, but, as discussed above, this can be largely identified and removed.

To identify other sources of systematic error, data were taken with the manipulator turned off (completely passive). If the manipulator links were frictionless and rigid, the phase of each component of the admittance should be -90 deg at all frequencies. In practice, however, there is friction, and the admittance shows a negative slope. At the highest measurement frequency (8 Hz), the phase of the diagonal terms of the admittance matrix is about -87 deg, while the phase of the off-diagonal terms is about -85 deg. In each term, the phase shows evidence of continuing to decrease with increasing frequency and, probably, of asymptotically approaching -90 deg. Nonetheless, these data cannot be used to reduce systematic phase errors to less than 5 deg.

The magnitude information in this data set is more useful. Although Bode plots of magnitude data show the correct shape (nearly constant slope of -20 dB/decade), the actual magnitudes are quite a bit higher than those that have been computed for the inertia matrix. The principal reason for the discrepancy is that the entire mass of the force transducer is included in the computation, while only that portion of its mass between the strain gauges and the base (about half the total) shows up in the measurement. The estimates of link inertias were also somewhat in error. In subsequent experiments, with feedback control, the controller designs have not been corrected for this error; this provides an opportunity to examine the robustness of these designs to modeling errors.

**Interaction Tests.** Data analysis for these experiments consist of transformation to endpoint coordinates and plotting.

### 6.2 Results

**Impedance Control.** The measured admittance and computed admittance of the impedance controlled manipulator are shown in Fig. 2. The two sets of plots compare well qualitatively, though they differ in a number of details. The greatest difference is in the amplitude of the admittance terms at low frequency: the measured values are considerably smaller. A probable explanation for this discrepancy is that the manipulator was tested out-of-plane by its connection to the spin. Even a small load at the endpoint will increase bearing friction at the joints substantially. At higher frequencies the match is better, and can be further improved by using the corrected inertia values found in the no control experiments.

The most important feature of Fig. 2 is that each plot lies entirely in the right half plane, indicating that, at least over the measured frequency range, the closed-loop admittance is passive. This conclusion (and, in fact, the conclusion that, over the entire frequency range, the admittance is passive) is supported by the results of the interaction tests which showed no signs of instability for any of the masses or springs tested.

**LQG/LTR Control.** Figure 7 shows the measured admittance of the manipulator with LQG/LTR control; it bears little resemblance to the computed admittance shown in Fig. 4. This difference is due to the fact that the LQG/LTR controller employs a model-based compensator, and is consequently much more sensitive to modeling errors than the impedance controller. Also shown in Fig. 7, however, is the predicted closed loop admittance when the inertial parameters of the plant are corrected (but the compensator is un-
changed). The qualitative agreement with the data is much better.

The measured admittance suggests that, in both the x and y directions, a minimum destabilizing mass and a minimum destabilizing stiffness will exist. The computed admittance further suggests that, in the y direction, a maximum destabilizing stiffness will exist. The interaction tests confirm the existence of such minimum masses and minimum and maximum stiffnesses in both the x and y directions. Representative results are shown in Figs. 8 and 9.

Contact instability data are shown in Fig. 10. Although discontinuous, contact instability correlates well to coupled instability. It stands to reason that a manipulator which is stable both in isolation and coupled to the environment will not exhibit contact instability. The act of contact does not itself create energy. Moreover, if a manipulator is unstable when coupled to the environment, sustained contact cannot be achieved. Thus, the correlation of contact and coupled stability is to be expected.

It is interesting to note that the contact instability data are qualitatively very similar to those that have been reported for force feedback controllers (Wlascich, 1986). Of course, force feedback is absent in this implementation. It is important to note also, that the contact instability of force-controlled manipulators may be examined with the same analytical tools used here (Colgate, 1989; Colgate and Hogan, 1989).

7 Conclusions

A necessary and sufficient condition for the stability of a linear, time-invariant multiport coupled to an arbitrary passive environment was presented. An approximate condition for the coupled stability of a multiport with discrete time control was also presented, and used to investigate two controllers designed for a simple manipulator. Two sets of experiments—measurement of closed-loop manipulator admittance and interaction with selected dynamic environments—corroborated the predictions of the analysis.

An important topic for future work is the development of similar, but less conservative coupled stability criteria that may be used when additional restrictions can be placed on the class of environments. For instance, if a frequency-dependent bound on the magnitude of the environment’s impedance is available in addition to the knowledge that it is passive, then there is certainly a set of manipulator behaviors (admittances) broader than the set of positive real admittances that will guarantee coupled stability. It is also important to exploit knowledge of structure. This is particularly true in multiport systems where some of the ports are physically distinct, as in telemanipulation (the telemanipulator couples to an operator and an environment at distinct locations). It has recently been shown that Doyle’s “structured singular value” (Doyle, 1981) can be used to develop the appropriate coupled stability criterion for such a case.
gate, 1991). A final topic for future research that should be mentioned is the development of approaches to controller design which will automatically satisfy coupled stability constraints.

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References


APPENDIX

Proof of the Multiport Coupled Stability Criterion

An additional version of the n-port passivity criterion is useful in the proof:

\[ n \text{-Port Passivity II. A linear time-invariant } n \text{-port is passive if:} \]

1. \[ H(s) \text{ has no poles in the open right half plane.} \]
2. \[ \text{The Nyquist plots of } H(s) + H^*(s) \text{ lie wholly in the closed right half plane.} \]

Note that criterion II is somewhat more intuitively appealing than criterion I given in Section 3, but is computationally more burdensome.

The most general interconnection of two multiport, linear time-invariant dynamic systems, \( H_1 \) and \( H_2 \), results in the closed loop characteristic equation:

\[ I + H_1(s)H_2(s) = 0 \]

The multivariable Nyquist criterion requires that (Maciejewski, 1989):

\[ \sum_{i=1}^{n} \Delta \text{arg det}(I + \lambda_i^{12}(s)) = -2\pi(P_c - P_e) \]

where \( \Delta \text{arg} \) refers to the change in argument as \( s \) traverses the Nyquist contour (indented as necessary to exclude imaginary poles), \( P_c \) is the number of poles of \( H_1(s) \) and \( H_2(s) \) in the open right half plane, and \( P_e \) is the number of poles of the coupled system in the open right half plane. This condition may be rewritten in terms of the eigenvalues, \( \lambda_i^{12}(s) \), of \( H_1(s)H_2(s) \):

\[ \sum_{i=1}^{n} \Delta \text{arg det}(I + \lambda_i^{12}(s)) = -2\pi(P_c - P_e) \]

The Nyquist plots of \( \lambda_i(s) \) are known as characteristic loci.

Using the characteristic loci, it is possible to state the multivariable Nyquist criterion in terms analogous to the classical, single loop criterion:

\[ P_e = P_c + \text{(total number of clockwise encirclements of the } -1 \text{ point by the characteristic loci)} \]

Because \( H_2(s) \) represents a passive system, it is positive real, which means, according to \( n \)-Port Passivity II, that it contains no poles in the open right half plane, and its characteristic loci lie completely within the closed right half plane.

 Sufficiency. It is obvious that the total energy of two passive systems, when coupled, must either remain constant or decrease; therefore the passivity of \( H_1 \) is sufficient to guarantee stability.

 Necessity. It is necessary to demonstrate that, if \( H_1(s) \) is not positive real, a passive \( H_2 \) exists that will destabilize the coupled system. Suppose that \( H_1(s) \) is stable but not positive real, and suppose that it has the eigen-decomposition:

\[ H_1(s) = W_1(s)\Lambda_1(s)V_1(s) \]

where \( W_1(s) \) is a matrix of left eigenvectors, \( \Lambda_1(s) \) is a
diagonal matrix of eigenvalues, and $V(s)$ is a matrix of right eigenvectors. Because $H_1(s)$ is not positive real, the Nyquist plot of at least one $\lambda_i^1(s)$, say $\lambda_i^1(s)$, must exhibit a clockwise-oriented segment in the left half plane (Colgate, 1988). We can imagine coupling $H_1(s)$ to a passive system with hybrid matrix $H_2(s)$ that has the same eigenstructure ($W_1(s)$ and $V_1(s)$) as $H_1(s)$, but with eigenvalues $\lambda_i^2(s)$ whose Nyquist plots lie wholly within the closed RHP. Because these systems have the same eigenstructure, it is obvious that, with proper ordering, $\lambda_i^{12}(s) = \lambda_i^1(s)\lambda_i^2(s)$. The problem has now been effectively reduced to that of 1-port interaction; following the original 1-port proof, $\lambda_i^2(s)$ can be selected to force an encirclement of the Nyquist plot of $\lambda_i^{12}(s)$ around the $-1$ point. This completes the proof.