1 Introduction

In an ideal transmission, one would expect both the ratio of output to input shaft speeds and the ratio of input to output applied torques to take on the same value as the transmission ratio setting. Further, the value of the speed ratio would have no dependence on the torque transmitted through the transmission and the value of the torque ratio would have no dependence on the speed at which the transmission runs. An ideal transmission would also consume no power. Although behavior approaching the ideal can generally be expected of transmissions realized using tractive rolling contacts. Some of the most promising covariances from the ideal can be expected of transmissions realized using tractive forces or simply traction mechanisms that rely on the support of either lateral or longitudinal forces across rolling contacts with spin. When a rolling contact between elastic bodies or even between rigid bodies in spin is called upon to transmit a tractive force, kinematic creep develops, expressing a departure from the intended rolling constraint. Creep in turn gives rise to nonideal properties in a cobot's virtual guiding surfaces. This paper develops simple models of the two transmissions by expressing the relative velocity field in the contact patch between rolling bodies in terms of creep and spin. Coulomb friction laws are applied in a quasi-static analysis to produce complete force-motion models. These models may be used to evaluate a cobot's ability to support forces against its virtual guiding surfaces. [DOI: 10.1115/1.1517560]
vides motive forces and the cobot directs motion using programmable motion constraints. Motion in only one instantaneously defined direction is allowed by the cobot. Using software control over the direction of the motion constraints, the cobot creates virtual fixtures within the shared workspace. The human can use these fixtures as guiding surfaces to develop superior manipulation strategies. To realize motion constraints at its end-effector, a cobot relies on a network of CVTs. The motion constraints are rendered programmable by active tuning of the CVT transmission ratios according to sensed human-applied force and sensed end-effector configuration.

There are two types of CVT used to construct cobots. The first is a simple steered wheel rolling on a planar surface. The steered wheel relates two linear speeds and supports forces acting in the direction of the wheel axel. The second type of CVT is the one treated in detail in this paper: the sphere-based CVT mentioned above. The sphere-based CVT relates two angular speeds and supports a certain ratio of input to output torques. Note that the wheel may be considered a translational CVT while the sphere-based CVT represents the more traditional application of the term CVT: as a rotational device.

Rolling contacts have been treated extensively in the literature. Models have been developed to describe the gradual breakdown and eventual failure of a rolling contact under applied longitudinal or lateral loads. The tangential traction and twisting moment have been expressed as functions of relative motion in a planar contact patch for conditions of fully developed sliding in both analytic [3] and computational models [4]. Howe and Cutkosky have applied such models to problems in sliding manipulation control [5]. Certain models account for elasticity in the wheel or rolling surface and account for the effects of spin or corning. Johnson [6] contains a particularly good review of rolling contact models.

Traction drive CVTs (as well as belt-drive CVTs) are being aggressively developed for application in automobiles, where they promise to increase fuel efficiency and driveability by eliminating gear shifting. Although these traction drive CVTs transmit tractive forces across rolling contacts like the CVT analyzed here, they also rely on the development of shear forces across a film of elastohydrodynamic oil that impinges between the rolling surfaces. The oil develops high viscosity under the high pressures in the rolling contact. This design has proven effective for the high torque transmission requirements of the automotive application. In contrast, the CVT analyzed in this paper uses dry friction between rolling bodies in direct contact. For a review of CVT designs for automotive application, see [7] and [8]. The CVT considered in this paper is very similar to the design developed for use in mobile robots described in [9] and the design developed for a nonholonomic manipulator described in [10]. A spherical rolling surface between a drive and driven wheel whose spin axis is rotated is common to all these designs. An analysis considering the mechanics of rolling contact in this CVT design has not been previously undertaken.

Our aim in this paper is to apply models of rolling contact to the CVT, to determine the manner in which traction-induced creep at each of the drive wheels expresses itself in deviations of the transmission law from the ideal. Our investigation is largely motivated by a suspected breakdown, for certain transmission ratio settings, of our CVT’s ability to maintain its speed constraint while supporting a load. When the rotational axis of the sphere passes through the contact patch of one of the drive rollers (a condition which occurs for transmission ratio values of zero or infinity), the slip in that roller’s contact patch is dominated by spin. In such case, one might intuitively expect that that drive roller is no longer capable of transmitting a longitudinal traction. A small torque load might then lead to loss of function (breakdown in the transmission law). To answer this question, we develop the simplest competent model of the CVT that may be used to relate nonideal performance characteristics to design parameters.

Prior to introducing the CVT, we discuss the steered wheel, which relates two linear speeds and force measure numbers rather than angular speeds and torque measure numbers. We introduce the ideal steered wheel and its function as a transmission in section 2. Section 3 introduces the CVT and presents its ideal kinematic and kinetic equations and establishes the analogy to the steered wheel. Section 4 introduces sideslip (lateral creep) in the steered wheel by referencing a simple model of an elastic Wheel. Section 5 presents a rigid body model of the CVT which features rigid drive rollers in tractive rolling with cornering. The nondimensionalized variables creep and spin are introduced and shown to be particularly advantageous descriptors of the kinematics of rolling since the tractive force may be simply expressed in terms of these variables. Finally, the full kinematic and kinetic equations are developed for the nonideal CVT. Deviations from the ideal are interpreted as a sideslip angle.
Express the velocity of the contact point \( C \) in basis \( N \) as \( \hat{N}_C = v_1 n_1 + v_2 n_2 \), where the scalars \( v_1 \) and \( v_2 \) are called the \( N \)-measure numbers of \( \hat{N}_C \). The ideal wheel allows motion of the contact point \( C \) in the longitudinal direction, yet prevents motion of \( C \) in the lateral direction: \( \hat{N}_C \cdot b_5 = 0 \). This stipulation may be expressed in the \( N \) basis, where it reads:

\[
\frac{v_2}{v_1} = \tan(\gamma) \quad (1)
\]

Thus the wheel can be viewed as a continuously variable transmission, setting the ratio of two translational speeds.

### 2.1 Force Balance

Let a force \( F \) applied to the wheel through its axle be expressed in the \( N \) basis as \( F = f_1 n_1 + f_2 n_2 \). The ideal wheel supports the lateral component of \( F \) with a friction force across the contact point \( C \). Yet the wheel accelerates in response to any longitudinal component of \( F \). For steady motion, the longitudinal component of \( F \) must be zero, which reads in basis \( N \):

\[
f_1 = f_2 = -\tan(\gamma) \quad (2)
\]

### 2.2 Coupling Space

To further elaborate on the function of the steered wheel as a continuously variable transmission, we introduce an abstract configuration space, which we label \( \Sigma \) and call coupling space. For the wheel, the axes of coupling space are associated with the linear displacements whose derivatives (speeds) are related by the transmission ratio. Thus the vectors spanning \( \Sigma \)-space, \( \sigma_1 \) and \( \sigma_2 \), are the same as \( n_1 \) and \( n_2 \), respectively.

We introduce a second basis \( U \) comprising unit vectors \( u_1 \) which defines the allowed direction and \( u_2 \) which defines the disallowed direction. In the case of the wheel, \( u_1 \) is parallel to \( b_1 \) and \( u_2 \) is parallel to \( b_3 \). The steered wheel allows motion in the \( u_1 \) direction yet resists motion in the \( u_2 \) direction. Conversely, the wheel supports forces applied in the \( u_2 \) direction, while steady motion requires that no force be applied in the \( u_1 \) direction. Thus we may state, regarding the \( U \)-measure numbers of the velocity of \( C \) and the force which may be supported across \( C \) in steady motion:

\[
v_1 = 0, \quad f_2 = 0 \quad (3)
\]

These statements, when rotated through the angle \( \gamma \) (re-expressed in the \( \Sigma \) basis) produce Eqs. (1) and (2).

In the case of the wheel, coupling space adds nothing new to the discussion. We introduce coupling space for the wheel in order to later draw analogies to the CVT (where \( \Sigma \) and \( U \) are non-trivially related to the bases describing the physical kinematics).

### 3 The Ideal Rotational CVT

Whereas the linear CVT (the steered wheel) employs one rolling contact, the rotational CVT employs four, making the kinematics of the rotational CVT significantly more complex. The construction of the rotational CVT involves a sphere of radius \( R \) in rolling contact with four cylindrical rollers, each of radius \( r \). Figure 3 shows two schematic views of the CVT. Rollers \( W1 \) and \( W2 \), called the drive rollers, appear with the sphere \( S \) in Fig. 3(a). The drive rollers have fixed, co-planar axes. Rollers \( R1 \) and \( R2 \), called the steering rollers, appear with \( S \) in Fig. 3(b). The steering rollers rotate freely about axes which may be oriented by steering. A reference basis \( A \) is fixed in the base of the CVT, and comprising three orthogonal unit vectors \( a_i \) \( (i = 1, 2, 3) \) is shown in both views (a) and (b). A second base-fixed basis \( N \) (seen in Fig. 3(a)) is established by rotating basis \( A \) about \( a_3 \) through \( -45^\circ \) while a third base-fixed basis \( B \) (seen in Fig. 3(b)) is established by rotating \( A \) about \( a_1 \) through \( +45^\circ \). The axis of steering roller \( R1 \), shown parallel to \( b_2 \), is oriented by rotation about \( -b_3 \) through the steering angle \( \phi \). Similarly, the axis of steering roller \( R2 \), shown parallel to \( b_1 \), is oriented by rotation about \( b_2 \) through the same steering angle \( \phi \). The steering angle settings of rollers \( R1 \) and \( R2 \) are coupled through a set of bevel gears not shown.

### 3.1 Kinematics of the Ideal CVT

The angular velocities in \( N \) of rollers \( W1 \) and \( W2 \) are denoted \( \hat{N}_C \cdot \omega_{W1} \) and \( \hat{N}_C \cdot \omega_{W2} \), respectively. Let the scalars \( \omega_1 \) and \( \omega_2 \) be defined according to \( \hat{N}_C \cdot \omega_{W1} = \omega_1 n_1 + \omega_2 n_2 \). Let the \( N \)-measure numbers of the angular velocity of \( S \) in \( N \) be defined as \( \hat{N}_C \cdot \Omega_1 n_1 + \Omega_2 n_2 + \Omega_3 n_3 \). A rolling constraint equation may be written for the contact between \( S \) and \( R1 \):

\[
\hat{N}_C \cdot \Omega_1 n_1 = \hat{N}_C \cdot \omega_{R1} \times -r b_3, \quad (4)
\]

while a second rolling constraint may be written for the contact between \( S \) and \( R2 \):

\[
\hat{N}_C \cdot \Omega_2 n_2 = \hat{N}_C \cdot \omega_{R2} \times -r b_3. \quad (5)
\]

Dot multiplication of these two vector equations with unit vectors \( n_1 \) \( (i = 1, 2, 3) \) produces six scalar equations which may be manipulated to yield

\[
\Omega_3 = 0 \quad (6)
\]

or the axis of rotation of sphere \( S \) lies in the \( n_1 \)–\( n_3 \) plane (which contains the axes of \( W1 \) and \( W2 \)). Secondly, the scalar equations yield

\[
\frac{\Omega_3}{\Omega_1} = \frac{\tan(\phi) - v_2}{\tan(\phi) + v_2} \quad (7)
\]

A rolling constraint may be written for each of the drive rollers, and these two equations divided to yield an expression for the transmission ratio:
as the angle subtended by the rotational axis of the idealized model, we assume that each rolling contact can transmit steady motion and balance internal forces in the CVT. In this way, relating the torques applied to the drive rollers, we may assume a transmission law relating angular speeds for the ideal CVT.

\[ \frac{\omega_2}{\omega_1} = -\frac{\Omega_2}{\Omega_1}, \]  

which together with Eq. (7) gives a formula for the transmission ratio in terms of the steering angle \( \phi \)

\[ \frac{\omega_2}{\omega_1} = -\tan(\phi) - \frac{v}{2} \]  

The ratio \( \Omega_2/\Omega_1 \), which is the common term between Eq. (7) and Eq. (8), orients the rotational axis of \( S \) in the \( n_1-n_3 \) plane. In summary, two rolling constraints at the steering rollers set this ratio, while an additional two ideal rolling constraints at the drive rollers relate this ratio to the ratio of speeds, \( \omega_1/\omega_2 \). We define \( \gamma \) as the angle subtended by the rotational axis of \( S \) (given by \( \theta = \Omega_1 n_1 + \Omega_3 n_3 \)) and the unit vector \( n_1 \), recognizing that

\[ \tan \gamma = -\frac{\Omega_3}{\Omega_1} \]  

Figure 4 explicitly shows the rotational axis of \( S \) drawn over the CVT view of Fig. 3(a) for an example \( \gamma \) value of about 30°. Note that trigonometry also reveals

\[ \frac{\omega_2}{\omega_1} = \frac{d_2}{d_1} = \tan(\gamma), \]  

where \( d_1 \) and \( d_2 \) are the perpendicular distance from each drive roller contact point to the rotational axis of \( S \). Equation (11) is the transmission law relating angular speeds for the ideal CVT.

3.1.2 Coupling Space. In Section 2, the abstract space \( \Sigma \), called coupling space, was introduced to illustrate the interpretation of the steered wheel as a transmission between two linear speeds. Likewise, a coupling space can be constructed for the rotational CVT and used to gain insight into both transmission laws: that relating speeds, Eq. (11) and that relating torques, Eq. (12). In their respective coupling spaces, the steered wheel and the CVT are completely analogous.

The unit vector \( \mathbf{t}_1 \) of the reference basis \( \Sigma \) is associated with angular displacements of drive roller \( W_1 \) and unit vector \( \mathbf{t}_2 \) is associated with angular displacements of drive roller \( W_2 \). Thus points in coupling space \( \Sigma \) correspond to various pairs of drive roller angular displacements while directions in coupling space are associated with various drive roller angular speed ratios. A reference basis \( U \) spanned by unit vectors \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) may be set up to describe the allowed and disallowed directions in coupling space. Basis \( U \) is oriented with respect to basis \( \Sigma \) by the angle \( \gamma \).

The angular speed \( \omega_1 \) of \( W_1 \) and the angular speed \( \omega_2 \) of \( W_2 \) are the \( \Sigma \)-measure numbers of a vector \( \mathbf{t}_1 \). \( \omega \) \( \omega_1 \mathbf{t}_1 + \omega_2 \mathbf{t}_2 \). After defining the \( U \)-measure numbers of \( \mathbf{t}_1 \) using \( \omega_3 \mathbf{u}_1 + \omega_4 \mathbf{u}_2 \), we see that the transmission law relating speeds, Eq. (11), requires

\[ \omega_3 = 0 \]  

A vector \( \mathbf{t} \) may be defined in coupling space to characterize the torques applied to the drive rollers \( W_1 \) and \( W_2 \). Define the \( \Sigma \)-measure numbers of \( \mathbf{t} \) using \( \tau \mathbf{t}_1 + \tau \mathbf{t}_2 \). Also define \( U \)-measure numbers of \( \mathbf{t} \) as follows: \( \tau \mathbf{u}_1 + \tau \mathbf{u}_2 \). Equation (12) reads in the \( U \) basis:

\[ \tau_1 = 0 \]  

The rotated bases \( \Sigma \) and \( U \) will prove very useful in discussions of the nonideal behavior of the CVT. The vectors \( \mathbf{t}_1 \) and \( \mathbf{t}_2 \) in coupling space encapsulate all functional aspects of the CVT. They are macro properties in the sense that they encompass the roller speeds or roller torques pairwise (rather than individually).

The magnitude of the vector \( \mathbf{t} \) is \( \tau \) by Eq. (14). It may be related to the torques \( \tau_1 \) and \( \tau_2 \) through:

\[ \tau_1 = -\tau_2 \sin(\gamma) \]  

\[ \tau_2 = \tau_2 \cos(\gamma) \]  

Under conditions of steady motion, the CVT may transmit a torque from one drive roller to the other. Traction forces at each drive roller develop to support this transmitted torque. In coupling space, the transmitted torque is interpreted as a torque \( \tau_1 \) in the disallowed direction. In the ideal transmission, any \( \tau_1 \) value may be transmitted without consequence to the transmission law relating angular speeds. This is not true in the non-ideal CVT, as we explore below.

4. The Nonideal Wheel

If a lateral load is applied during rolling, the wheel will drift laterally as it rolls ahead. Its actual velocity \( \mathbf{v} \) will make an angle \( \alpha \), known as the sideslip angle, with the wheel heading \( \mathbf{h}_1 \). Lateral creep \( \xi_{xy} \), defined as the ratio of the lateral speed \( v_{y} \) to the absolute value of the rolling speed \( v_1 \), is related to the sideslip angle as follows:

\[ \xi_{xy} = \frac{v_y}{|v_1|} = \tan(\alpha). \]  

A relationship between sideslip angle \( \alpha \) and applied lateral force \( F_y \) may be constructed by considering radial and axial deformations in an elastic wheel that give rise to a curvilinear contact line against the rolling surface comprising a linear sticking region and a curving slipping region. The sideslip angle dictates the direction of the linear region with respect to the heading while in the slipping region, the accumulated stresses relax. Across the sticking region, static friction will apply while across the slipping region, kinetic friction will apply. Integrating across the entire contact line, one arrives at an expression for the lateral force. One
such relationship (see [6]) is shown in Fig. 5. The tangent of the sideslip angle, normalized by $R/\mu a$, is plotted versus the non-dimensionalized lateral force $F_{\perp}/\mu P$, where $R$ is the wheel radius, $a$ is the contact line half-width, $\mu$ is the coefficient of friction, and $P$ is the normal force. Note that for small sideslip angles, the relationship between sideslip angle and disallowed force is approximately linear. The slope $C$ of the linear portion is called cornering stiffness in tire mechanics. Past a certain sideslip angle, the wheel’s ability to support lateral forces breaks down until the entire contact line is in slip.

5 The Nonideal CVT

Like the wheel, a physically realizable CVT cannot strictly prevent motion in the disallowed direction. Under a disallowed force, the rolling constraints at the drive rollers give way to creep. In contrast to the Wheel, where lateral traction supports a disallowed force, in the CVT it is *longitudinal* traction that supports a disallowed force. Moreover, there are two longitudinal tractions in the CVT: one at each drive roller. Associated with this pair of longitudinal tractions is a pair of longitudinal creeps. In coupling space, this pair of longitudinal creeps may be interpreted as a single lateral creep or expressed as a sideslip angle. But before we consider the longitudinal creeps as a pair in coupling space, we must consider them individually in physical space.

In contrast to lateral traction and creep, an elastic model is not required to express longitudinal creep as a function of longitudinal traction so long as two conditions are met. First, spin must be present. That is, the two rolling bodies must have a relative angular velocity with non-zero component in the contact normal direction. Second, the two rolling bodies must contact at more than a single point. Both of these conditions are met in a rigid cylindrical wheel turning a corner while rolling on a plane. Both conditions are also met in a rigid cylindrical wheel rolling on a cone.

When spin is nonzero, there is only one point of the contact patch in rolling; the remaining portions of the contact patch must be in sliding. Sliding is in one direction on one side of the rolling point and in the other direction on the other side. After representing the sliding in a relative velocity field, the traction transmitted across the contact patch can be computed as the vector sum of the friction force acting at each point in the contact patch. If a braking axial torque is applied (balanced by traction on the wheel opposite the direction of rolling,) the point of rolling will migrate toward the center of rotation, whereas if an accelerating axial torque is applied (balanced by traction on the wheel in the direction of rolling,) the rolling point will migrate away from the center of rotation. Any deviation of the rolling point away from the center of the contact line produces a longitudinal creep.

In the following, we will construct a rigid body model of the CVT that accounts for longitudinal creep in each of the drive rollers. The drive rollers of the CVT meet the first condition for creep, since spin is indeed present in the contact patch between each drive roller and the sphere. Each drive roller is in a state of cornering, with the cornering radius established by the transmission ratio. In fact (see Fig. 4), as the transmission ratio approaches zero, the axis of rotation of the sphere approaches the center of one of the contact patches, causing the cornering radius for that roller to approach zero and the cornering radius of the other roller to approach infinity. As the transmission ratio approaches infinity, the situation for the two drive rollers is reversed. Insofar that creep is a function of cornering radius, we expect to see a dependence of the CVT transmission law for speeds on the transmitted load. Further, we expect that this dependence will vary as a function of the steering angle setting (the transmission ratio setting) and wish to quantify this dependence.

The second requirement for the development of creep in rolling rigid bodies is met if the bodies make fine contact. Line contact is not possible between rigid cylindrical rollers and a rigid sphere, so we modify the geometric model of the CVT as described in section 5.1.

5.1 Kinematics of the Nonideal CVT. Line contact is set up between the sphere and each of the rollers by modeling the sphere as a pair of connected co-axial cones. Figure 6 shows the system $S$ comprising two cone sections. The vertex angle of each cone is chosen so that line contact is established between $S$ and drive rollers $W1$ and $W2$. Note that the vertex angles vary as $\gamma$ varies. A reference basis $T$ of unit vectors $\mathbf{t}_i$ ($i=1,2,3$) is used to locate the rotational axis of $S$ as follows: basis $T$ is first aligned with $N$ and then subjected to a right-hand rotation about $n_3$ through the angle $\gamma$. System $S$ rotates with an angular speed $\Omega$ about unit vector $\mathbf{t}_1$, or $\dot{\mathbf{n}}^T = \Omega \mathbf{t}_1$. Drive rollers $W1$ and $W2$,
both of radius \( r \) and width \( 2a \), roll on the cones of \( S \). As before, the axis of drive roller \( W_1 \) is aligned with \( n_1 \) and the axis of drive roller \( W_2 \) is aligned with \( n_2 \).

The angle \( \gamma \) is set by the steering angle \( \phi \) of the steering rollers according to Eq. (9). We assume that the steering rollers are able to maintain the orientation of the axis of the sphere, without dependence on the traction forces developed at the drive rollers. This assumption is based on the fact that the traction forces at the drive rollers have no moment about \( n_2 \), and therefore no influence over the rotation of basis \( T \) (which locates the rotational axis of \( S \)) about \( n_2 \).

Define the center of each contact patch as point \( O_i \) \( (i = 1, 2) \). Additionally, as shown in Fig. 6, a generic point \( P \), is located a distance \( x \) from \( O_i \) and a special point \( Q_i \), is located a distance \( \delta_i \) away from \( O_i \) \( (i = 1, 2) \). Point \( Q_j \) \( (i = 1, 2) \), called the rolling point, is used to identify the unique position in each contact patch at which the relative velocity is zero. Measure numbers for the angular velocity \( N \omega^S \) of \( S \) in \( N \), the angular velocity \( N \omega^{W_1} \) of \( W_1 \) in \( N \), and the angular velocity \( N \omega^{W_2} \) of \( W_2 \) in \( N \) are defined as follows:

\[
N \omega^S = \Omega \Omega_1 \quad N \omega^{W_1} = - \omega_1 n_1 \quad N \omega^{W_2} = \omega_2 n_1
\]

(17)

Let us first develop the relative velocity field for each contact line in terms of creep and spin. Subsequently, we will find the resultant tractive force by application of the Coulomb law and integration. Additionally, we will find the traction force by integration. We begin with the contact line between \( W_1 \) and \( S \).

The relative velocity at the generic point \( P \) may be expressed

\[
\Delta v^P = \omega^{W_1} \times (r n_1 + x_1 n_1) - \omega^S \times (-r n_1 + x_1 n_1).
\]

(18)

After rearranging terms and defining the relative velocity at \( O \) as \( \Delta v^O \) and the relative angular velocity between \( W_1 \) and \( S \) as \( \Delta \omega_1 = \omega^{W_1} - \omega^S \), one finds

\[
\Delta v^P = \Delta v^O + \Delta \omega_1 \times x_1 n_1
\]

(19)

That is, the relative velocity field in the contact patch may be characterized as a rigid body motion. A similar vector expression holds for the contact patch between \( W_2 \) and \( S \). Once the cross product has been carried out, all vectors are in the \( n_2 \) direction, and the subscripts 1 or 2 may be dropped.

After defining \( \Delta \omega = \Delta \omega n_1 \), and noting that \( \Delta v^P \) and \( \Delta v^O \) have only \( n_2 \) measure numbers (which are denoted without boldface), we have:

\[
\Delta v^P \cdot n_2 = \Delta v^O \cdot n_2 + \Delta \omega \times x \cdot n_2.
\]

(20)

To nondimensionalize the terms in this equation, the absolute value of the common speed \( |v^P| \) at \( Q \) is chosen, where \( Q \) is the rolling point. Equation (20) may be dot multiplied by \( n_2 \) and divided through by \( |v^P| \) to yield:

\[
\frac{\Delta v^P}{|v^P|} = \frac{\Delta v^O}{|v^P|} + \frac{\Delta \omega \alpha}{|v^P|} \times \frac{x}{\alpha}
\]

(21)

The left hand side we define as the slip \( \lambda(x) \) at the point \( P \) of the contact patch. The first term on the right hand side we define as longitudinal creep \( \xi \) and the factor \( \Delta \omega \alpha / |v^P| \) in the last term we define as spin \( \psi \). Recall that \( \alpha \) is the contact line half-width. We now have an expression for the slip at a generic point in the contact patch in terms of the bulk properties creep \( \xi \) and spin \( \psi \):

\[
\lambda(x) = \xi + \psi \frac{x}{\alpha}
\]

(22)

This expression describes the slip in the contact patch between \( W_1 \) and \( S \) and between \( W_2 \) and \( S \). To particularize the expression for each contact patch, the subscripts 1 and 2 may be used with \( \xi \), \( \psi \), \( \lambda \), and \( x \).

We now derive the creep and spin parameters for each roller. A rolling constraint at the point \( Q_1 \) produces the constraint equation

\[
\omega_1 r = \Omega (\cos(\gamma)R - \sin(\gamma)\delta_1).
\]

(23)

Applying the definitions of \( \xi \) and \( \psi \) and making use of Eq. (23), we have

\[
\xi_1 = \frac{-\delta_1 \sin(\gamma)}{|R \cos(\gamma) - \delta_1 \sin(\gamma)|}
\]

(24)

\[
\psi_1 = \frac{\sin(\gamma)\alpha}{|R \cos(\gamma) - \delta_1 \sin(\gamma)|}
\]

(25)

Likewise, a rolling constraint at the point \( Q_2 \) produces

\[
\omega_2 r = \Omega (\sin(\gamma)R - \cos(\gamma)\delta_2).
\]

(26)

The creep and spin at roller 2 in terms of \( \gamma \) and \( \delta_2 \) are

\[
\xi_2 = \frac{-\delta_2 \cos(\gamma)}{|R \sin(\gamma) - \delta_2 \cos(\gamma)|}
\]

(27)

\[
\psi_2 = \frac{-\cos(\gamma)\alpha}{|R \sin(\gamma) - \delta_2 \cos(\gamma)|}
\]

(28)

Dividing Eqs. (26) and (23), we have

\[
\frac{\omega_1}{\omega_2} = \frac{\cos(\gamma) - \frac{1}{R} \delta_1 \sin(\gamma)}{\sin(\gamma) - \frac{1}{R} \delta_2 \cos(\gamma)}.
\]

(29)

The variables \( \delta_1 \) and \( \delta_2 \) may be replaced by ratios of the creep and spin parameters using Eqs. (24), (25), (27), and (28), giving:

\[
\frac{\omega_1}{\omega_2} = \frac{\cos(\gamma) + \frac{\alpha \xi_1}{R \psi_1} \sin(\gamma)}{\sin(\gamma) - \frac{\alpha \xi_2}{R \psi_2} \cos(\gamma)}.
\]

(30)

Equation (30) is the non-ideal transmission law relating angular speeds. It expresses the ratio of angular speeds as a function of the creep to spin ratios at each drive roller, which in turn are functions of the tractions transmitted across each contact patch. We are now ready to develop the dependence of creep and spin on the tractive forces.

5.2 Force Balance for the Non-ideal CVT. In section 3.1.1, we assumed that any tractive force could be transmitted across the contacts between drive rollers and sphere, without consequence to the rolling constraints. This assumption no longer holds in the present section, where we treat the nonideal CVT. Now the tractive force affects the rolling constraints, giving rise first (at low magnitudes) to longitudinal creep and eventually to gross longitudinal slip.

Our present objective is to relate the creep and spin parameters to the tractive force at each contact patch. This can be done by appealing to Eq. (22) and applying the law of Coulomb friction. Equation (22) gives the slip at a point located by \( x \) in a contact patch whose velocity field is characterized by \( \xi \) and \( \psi \). Note that \( \lambda(x) \) (the slip at \( P \)) and \( \Delta v^P \) (the relative velocity at \( P \)) have the same sign, so \( \lambda \) may be used to form a vector expression for the friction force acting on the sphere (from the roller) at point \( P \), whose direction may be expressed with respect to a fixed reference basis. The friction force acting on \( S \) across a differential element of the contact patch is given by the Coulomb law:

\[
dF = -\mu \frac{N}{2a} \text{sgn}(\lambda(x)) \cdot n_2
\]

(31)
Integrating across the contact patch, the normalized tractive force $F/N$ is plotted against $\psi$ in terms of creep $\xi$ and spin $\psi$ in Fig. 7. The non-dimensionalized tractive force $F/\mu N$ is plotted against $\xi$ and $\psi$ in Fig. 7. The non-dimensionalized tractive torque $M/a \mu N$ is plotted against $\xi$ and $\psi$ in Fig. 8. Each of the four regions, divided by the planes $\xi = \psi$ and $\xi = -\psi$, have been noted in Figs. 7 and 8.

Table 1 gives the desired relationship between the creep and spin parameters and the tractive force. Cases I and II account for creep whereas cases III and IV describe limiting friction (gross slip). Although it was constructed by expressing the tractive force in terms of $\xi$ and $\psi$, Table 1 can also be used in the reverse direction, to express the ratio of $\xi$ to $\psi$ in terms of the tractive force. This is how Table 1 will be used in our study of the kinetics of the CVT in the following section.

<table>
<thead>
<tr>
<th>case</th>
<th>limits</th>
<th>$\xi + \psi &gt; 0$</th>
<th>$\xi - \psi &lt; 0$</th>
<th>$\xi + \psi &lt; 0$</th>
<th>$\xi - \psi &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$\psi$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
</tr>
<tr>
<td>II</td>
<td>$\psi$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
</tr>
<tr>
<td>III</td>
<td>$\psi$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
</tr>
<tr>
<td>IV</td>
<td>$\psi$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
<td>$\frac{1}{2} \left( \frac{1}{(\psi)^2} \right)$</td>
</tr>
</tbody>
</table>

Note that in this discussion we have neglected the contribution of the traction torques to the moment balance on $S$. The contribution of the traction torques is quite small in comparison to that of the tractive forces, especially when the contact patches are small. 

5.3 Kinetics of the Nonideal CVT. We are now ready to couple the transmission law relating angular speeds, Eq. (30), to the transmission law relating angular speed numbers, Eq. (34), through the application of Table 1. The only complication is presented by the four cases of Table 1 that account for limiting friction. The situation can be handled by considering $\gamma$ in increments of $\pi/4$, for the quadrant in which $\gamma$ falls determines the sign of $\psi_1$ and $\psi_2$, which allows one to differentiate cases I and II in Table 1. The boundaries of cases III and IV can be differentiated if $\gamma$ is further subdivided into sectors of $\pi/8$.

Figure 9 shows the governing relationships, laid out graphically around a circle to indicate the range of $\gamma$ in which they are valid. The sign of $\psi_i$ ($i = 1, 2$), indicated in the innermost ring of Fig. 9, is constructed with reference to Eqs. (25) and (28). The relationship between the tractive torque and the creep to spin ratio is constructed with reference to Table 1 and given in the intermediate ring of Fig. 9. Considering $\gamma$ in increments of $\pi/4$, the normalized torques can be substituted for the creep to spin ratios in Eq. (30). However, having identified $W_2$ as the drive and $W_1$ as the driven wheel, we must incorporate a dependence on $\gamma$ of the sign on the
Fig. 9 The governing relationships are drawn graphically around a circle so as to indicate the range of \( \gamma \) in which they are valid. In concentric regions starting at the center, the signs of the spin parameters, the relationships between the creep to spin ratios and the drive roller torques, and the drive roller torque values at limiting friction are noted.

Second term in both the numerator and denominator of Eq. (30) to ensure that the CVT acts as a dissipative element:

\[
\frac{\omega_1}{\omega_2} = \frac{\cos(\gamma) + \text{sgn}(\tan(\gamma)) \frac{a}{R} \frac{\xi_1}{\psi_1} \sin(\gamma)}{\sin(\gamma) - \text{sgn}(\tan(\gamma)) \frac{a}{R} \frac{\xi_2}{\psi_2} \cos(\gamma)}.
\] (35)

This dependence will ensure that the projections of \( \boldsymbol{\omega} = \omega_1 \mathbf{u}_1 + \omega_2 \mathbf{u}_2 \) and \( \tau = \tau_1 \mathbf{u}_1 + \tau_2 \mathbf{u}_2 \) obey the law: \( \text{sgn}(\omega_1 \tau_1) = -\text{sgn}(\omega_2 \tau_2) \), which ensures that power to the CVT flows in opposite directions on either drive roller. In effect, we have reversed the direction of \( \tau \) when \( \mathbf{u}_1 \) has positive directions in quadrants II and III of \( \Sigma \)-space and reversed the direction of \( \boldsymbol{\omega} \) when \( \mathbf{u}_2 \) has positive directions quadrants III and IV. This allows us to employ our analysis that made use of the rotation of the \( U \) basis relative to the \( \Sigma \) basis to describe our CVT (consistent with \( \omega_1 / \omega_2 = 1/\tan(\gamma) \), \( \tau_2 / \tau_1 = -1/\tan(\gamma) \)), while constructing relationships that refer to the traditional input-output identified transmission laws \( \omega_1 / \omega_2 = \tau_2 / \tau_1 = 1/\tan(\gamma) \) when evaluating its performance.

So, considering \( \gamma \) in increments of \( \pi/2 \), the non-dimensionalized torques can be substituted for the creep to spin ratios in Eq. (35). Then Eq. (34) can be used to obtain an expression for the speed ratio \( \omega_1 / \omega_2 \) in terms of the torque load \( \tau_{\perp} \). For example, for \( 0 < \gamma < \pi/2 \):

\[
\frac{\omega_1}{\omega_2} = \frac{\cos(\gamma) - \frac{a}{R} \frac{\tau_{\perp}}{r \mu N} \sin^2(\gamma)}{\sin(\gamma) + \frac{a}{R} \frac{\tau_{\perp}}{r \mu N} \cos^2(\gamma)}.
\] (36)

For \( -\pi/2 < \gamma < 0 \),

\[
\frac{\omega_1}{\omega_2} = \frac{\cos(\gamma) + \frac{a}{R} \frac{\tau_{\perp}}{r \mu N} \sin^2(\gamma)}{\sin(\gamma) - \frac{a}{R} \frac{\tau_{\perp}}{r \mu N} \cos^2(\gamma)}.
\] (38)

These expressions hold so long as the slip is not all in the same direction, in either contact line.

Beyond a certain \( \tau_{\perp} \), which will itself be a function of \( \gamma \), there will be no more available tractive force in one or the other drive rollers. At that time, the normalized tractive force at that roller will be unity, the slip will be in the same direction across the entire contact patch, and equivalently, the rolling point \( Q \) will be at the edge of that contact patch. With a disallowed force \( \tau_{\perp} \) which increases beyond that point, the sideslip angle will increase unchecked and there will be acceleration in the disallowed direction. To find the limits of performance, the creep and spin variables are particularly handy, since the borderline cases are symmetric in \( \xi - \psi \) space. Let us call the traction associated with slip all in the same direction within a contact line “limiting traction.”

Whether limiting traction is attained first at roller \( W_1 \) or \( W_2 \) depends on the angle \( \gamma \). Since \( \cos(\gamma) > \sin(\gamma) \) for \( 0 < \gamma < \pi/4 \), by Eq. (34) roller \( W_2 \) will first attain the limit \( \tau_{\perp} / r \mu N = 1 \) and by Eq. (34), \( \tau_{\perp} / r \mu N = -\tan(\gamma) \), as noted in the outermost ring in Figure 9. Together with the traction torque and creep to spin ratio relationships for \( 0 < \gamma < \pi/2 \) (Eq. (35) and the intermediate ring of Table 9), these limiting values give

\[
\frac{\omega_1}{\omega_2} = \frac{c^2 - \frac{a}{R} s^2 \gamma}{s \gamma + \frac{a}{R} c^2 \gamma}.
\] (38)

where \( c \gamma \) denotes \( \cos(\gamma) \) and \( s \gamma \) denotes \( \sin(\gamma) \). For \( \pi/4 < \gamma < \pi/2 \), roller \( W_1 \) first attains the limit and

\[
\frac{\omega_1}{\omega_2} = \frac{s \gamma - \frac{a}{R} c^2 \gamma}{s^2 \gamma + \frac{a}{R} c^2 \gamma}.
\] (39)

Analogous considerations lead to similar expressions for the remaining \( \pi/4 \) sectors of \( \gamma \). Figure 10 shows the speed ratio \( \omega_1 / \omega_2 \) at limiting traction as a function of the CVT steering angle \( \gamma \) when
such as elasticity would contribute to a softening of the transition

\[ \Delta \tau_{1} \] for \( \gamma \) close to \( \pi / 2 \). The remaining two quadrants of \( \gamma \) are essentially copies of this graph with similar discontinuities at \( \gamma = n \pi / 2 \), \( n = 0, 1, 2, \ldots \). Equations (42) and (43) were used to plot the upward sloping surface. Eqs. (38) and (39) with Eq. (41) were used to locate the limiting friction curve, shown as a curved vertical fence in Fig. 11.

This figure shows neatly that \( \alpha \) is a nearly linear function of \( \tau_{1} \) with a slope that is a weak function of \( \gamma \). The highest value of \( \tau_{1} \) attained before gross slip occurs is a function of \( \gamma \) with minima of \( \tau_{1} \) at \( \gamma = n \pi / 2 \) and maxima of \( \tau_{1} \) at \( \gamma = 4 \pi / 4 + n \pi / 2 \), \( n = 0, 1, 2, 3 \). Values of \( \alpha \) increase with increasing \( \tau_{1} \) for \( 0 < \gamma < \pi / 2 \) but decrease with increasing \( \tau_{1} \) for \( - \pi / 2 < \gamma < 0 \), consistent with the dissipative property of the CVT.

As anticipated by intuition, the sideslip for a given \( \tau_{1} \) is greater at \( \gamma = n \pi / 2 \), \( n = 0, 1, 2, \ldots \) when the rotational axis of \( S \) intersects one of the contact lines and the limiting value of \( \tau_{1} \) is lower. Further, at these transmission ratios, the sideslip angle undergoes a reversal in sign to account for the power sign convention (the changing roles of drive and driven rollers that ensure dissipativity). Indeed the feared breakdown that intuition had suggested does exist. Null and infinity transmission ratios are not achievable in the non-ideal case.

6 Summary

In this paper, we have developed the kinetics of the wheel and the CVT, illustrating the mechanism in each device by which the transmission law relating speeds becomes coupled to that relating forces. To describe deviation from the ideal rolling constraint, we used kinematic creep. Viewed in coupling space, both the wheel and the CVT are subject to lateral creep when a force is applied in the disallowed direction. In physical space, however, the wheel supports a disallowed force with lateral traction and is subject to lateral creep, whereas the CVT supports a disallowed force (pair of drive roller torques) with a pair of longitudinal tractions and is subject to a pair of longitudinal creeps. The fact that the tractions are transmitted across contacts that undergo spin whose magnitude is again related to the transmission ratio makes the longitudinal creeps transmission ratio dependent in addition to load dependent. The pair of longitudinal creeps in the CVT, when viewed in coupling space, is interpreted as lateral creep. To express lateral creep in the wheel as a function of lateral traction, an elastic model is used. To express the longitudinal creeps in the CVT as a function of longitudinal tractions, a rigid body model suffices, so long as the rolling bodies make line contact. An approximating physical model (two coaxial cones in place of the spherical rolling surface) was used to set up line contact.

Analytical expressions relating tractive force to creep and spin at each of two line contacts were incorporated into a full kinetostatic model of the CVT to yield the dependence of the speed ratio on the transmitted load and transmission ratio setting. Deviations from the ideal rendered very high transmission ratios unattainable and very low difficult to regulate. However, an approximately
linear relationship between sideslip angle and transmitted load is confirmed for the full range of transmission ratio settings.

These wheel and CVT models may be used to create model-based controllers for cobots that compensate for the effects of sideslip by counter-steering. Sideslip in the CVT causes a guiding surface to give way as an operator pushes against it. Counter-steering as a function of transmission ratio setting and applied load would restore the guiding surface. Although, according to the models presented here, only limited compensation may be provided, especially in the regions of very high or very low transmission ratios. Extensions to the models presented here include the addition of the elastic features of the Wheel model to the CVT model. In such an elastic model, the cones replacing the sphere will be unnecessary, and the effects of an elliptical contact patch may be explored.

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References